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BOUNDARY VALUES IN CHROMATIC GRAPH THEORY

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Let G be a planar graph drawn in the plane so that its outer boundary Γ is a k-cycle. A four-coloring of Γ is *admissible* if it extends to a four-coloring of all of G. Let ψ be the number of admissible boundary colorings, and we suppose the truth of the Four-Color Conjecture in the theorems marked with a * below.

CONJECTURE. $\psi \geqq 3 \cdot 2^k$ ($k = 3, 4, \ldots$). (The sign of equality holds if G is a triangulation of a k-cycle with no interior vertices.)

*THEOREM 1. *$\psi \geqq 24F_{k-1} \geqq C((1 + 5^{1/2})/2)^k$, where F_k is the kth Fibonacci number.*

*THEOREM 2. *$\psi \geqq 3 \cdot 2^k$ for $k = 3, 4, 5, 6$.*

A graph is *totally reducible* (t.r.) if every four-coloring of the boundary is admissible (i.e., $\psi = 3^k + (-1)^k \cdot 3$).

THEOREM 3. *For each k there is a t.r. graph G whose boundary is a k-cycle and whose interior is a triangulation.*

An *annulus* G_{kl} is an l-cycle drawn interior to a k-cycle, with a maximum number of nonintersecting edges connecting the two cycles. The vertices of the l-cycle are $u_1, u_2, \ldots, u_l$, and $\rho(u)$ is the valence of the vertex u.

THEOREM 4. *An annulus G_{kl} is t.r. iff it has none of the following properties*: (1) $\rho(u_1) \geqq 6$; (2) $\rho(u_i) = \rho(u_j) = 5$ $(j \leqq k - 3)$ *and* $\rho(u_i) = 4$ *for all* i *in* $1 < i < j$; (3) $\rho(u_1) = \rho(u_j) = 5$, $\rho(u_i) = 4$ *for all* i *in* $1 < i < j$, $j = k - 2$, k *even*; (4) $\rho(u_1) = 5$, $\rho(u_i) = 4$ *for all* $1 < i < l$, l *odd*.

*THEOREM 5. *An annulus G_{kl} satisfies the Conjecture stated above.*

Proofs will appear elsewhere.

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