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THE L^p-INTEGRABILITY OF THE PARTIAL DERIVATIVES OF A QUASICONFORMAL MAPPING

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ABSTRACT. Suppose that $f: D \to R^n$ is an n-dimensional K-quasiconformal mapping. Then the first partial derivatives of f are locally L^p-integrable in D for $p \in [1, n + c)$, where c is a positive constant which depends only on K and n.

Suppose that D is a domain in Euclidean n-space R^n where $n \geqq 2$, and that $f: D \to R^n$ is a homeomorphism. For each $x \in D$ we let

$$L_f(x) = \limsup_{y \to x} |f(y) - f(x)|/|y - x|,$$

$$J_f(x) = \limsup_{r \to 0} m(f(B(x, r)))/m(B(x, r)),$$

where $B(x, r)$ denotes the open n-dimensional ball of radius r about x and m denotes Lebesgue measure in R^n. We call $L_f(x)$ and $J_f(x)$, respectively, the maximum stretching and generalized Jacobian for the homeomorphism f at the point x. These functions are nonnegative and measurable in D, and Lebesgue's theorem implies that J_f is locally L^1-integrable there.

Suppose in addition that f is K-quasiconformal in D. Then $L_f^n \leqq KJ_f$ a.e. in D, and thus L_f is locally L^n-integrable in D. Bojarski has shown in [**1**] that a little more is true in the case where $n = 2$, namely that L_f is locally L^p-integrable in D for $p \in [2, 2 + c)$, where c is a positive constant which depends only on K. His proof consists of applying the Calderón-Zygmund inequality [**2**] to the Hilbert transform which relates the complex derivatives of a normalized plane quasiconformal mapping. Unfortunately this elegant two-dimensional argument does not suggest what the situation is when $n > 2$.

The purpose of this note is to announce the following n-dimensional version of Bojarski's theorem.

THEOREM. *Suppose that D is a domain in R^n and that $f: D \to R^n$ is a K-quasiconformal mapping. Then L_f is locally L^p-integrable in D for $p \in [1, n + c)$, where c is a positive constant which depends only on K and n.*

This result is derived from the following two lemmas. The first is an inequality relating the L^1- and L^n-means of L_f over small n-cubes, while

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the second derives the integrability from this inequality.

LEMMA 1. *Suppose that D is a domain in R^n, that $f : D \to R^n$ is a K-quasiconformal mapping, and that Q is a closed n-cube in D with*

$$\operatorname{dia} f(Q) < \operatorname{dist}(f(Q), \partial f(D)).$$

Then

$$\frac{1}{m(Q)} \int_Q L_f^n \, dm \leqq b \left(\frac{1}{m(Q)} \int_Q L_f \, dm \right)^n,$$

where b is a constant which depends only on K and n.

LEMMA 2. *Suppose that $q, b \in (1, \infty)$, that Q is a closed n-cube in R^n, that $g : Q \to [0, \infty]$ is L^1-integrable in Q, and that*

$$\frac{1}{m(Q')} \int_{Q'} g^q \, dm \leqq b \left(\frac{1}{m(Q')} \int_{Q'} g \, dm \right)^q$$

for each parallel closed n-cube $Q' \subset Q$. Then g is L^p-integrable in Q for $p \in [1, q + c)$, where c is a positive constant which depends only on q, b and n.

Complete proofs for these results will appear shortly in [**3**].

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