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PROJECTIVE MODULES FOR FINITE CHEVALLEY GROUPS

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1. **Introduction.** The irreducible modular representations of the finite Chevalley groups (and their twisted analogues) have been described by C. W. Curtis and R. Steinberg (see [**1**], [**2**], [**8**]). In this note we outline some parallel results on principal indecomposable modules (PIM's), for which proofs will appear elsewhere. The groups $\mathbf{SL}(2, q)$ are already treated in detail in [**5**], based in part on the method of A.V. Jeyakumar [**6**].

K denotes an algebraically closed field of prime characteristic p, over which all modules are assumed to be finite dimensional. Our notation resembles that of [**4**], [**5**]: $\mathbf{G}$ is a simply connected algebraic group (of simple type), $\mathfrak{g}$ its Lie algebra, $\mathscr{U}$ the restricted universal enveloping algebra of $\mathfrak{g}$, G_q the group of rational points of $\mathbf{G}$ over a field of q elements, $\mathscr{R}_q$ the group algebra of G_q over K. (When $q = p$, we write simply $G, \mathscr{R}$.)

EXAMPLE. $\mathbf{G} = \mathbf{SL}(2, K)$, $\mathfrak{g} = \mathfrak{sl}(2, K)$, $G_q = \mathbf{SL}(2, q)$.

The set Λ of restricted highest weights (determined by integers between 0 and $p - 1$) indexes the (classes of) irreducible modules M_λ for $\mathscr{U}$ (or $\mathscr{R}$). If $\lambda = \lambda_0 + \lambda_1 p + \cdots + \lambda_k p^k$ ($\lambda_i \in \Lambda$), then the twisted tensor product modules $M_\lambda = M_{\lambda_0} \otimes M_{\lambda_1}^{(p)} \otimes \cdots \otimes M_{\lambda_k}^{(p^k)}$ exhaust the (classes of) irreducible modules for $\mathscr{R}_q$ ($q = p^{k+1}$) and for $\mathbf{G}$ (as k runs over all nonnegative integers). Denote by $U_\lambda, R_\lambda, R_{\boldsymbol{\lambda}}$ the respective PIM of $\mathscr{U}, \mathscr{R}, \mathscr{R}_q$ having top composition factor $M_\lambda, M_\lambda, M_{\boldsymbol{\lambda}}$. The only irreducible module which is also projective is the Steinberg module $M_\sigma = U_\sigma = R_\sigma$, $\sigma = (p-1)\delta$, $\delta =$ half-sum of positive roots. A similar statement is true for $M_{\boldsymbol{\sigma}} = R_{\boldsymbol{\sigma}}$ ($\boldsymbol{\sigma} = \sigma + \sigma p + \cdots + \sigma p^k$).

2. **Projective modules.**

LEMMA. *Let V, W be modules for the restricted universal enveloping algebra of a restricted Lie algebra, with W projective. Then $V \otimes W$ is also projective.*

This is proved in [**7**]. The analogous statement for the group algebra of a finite group is well known [**3**, Exercise 2, p. 426].

We apply the lemma as follows. For $\mu \in \Lambda$, define $T_\mu = M_\mu \otimes M_\sigma$ (σ as above). This is a module for $\mathbf{G}, \mathscr{R}, \mathscr{U}$, and is projective for $\mathscr{R}, \mathscr{U}$ (since M_σ is). In particular, T_μ is the direct sum of certain $\mathscr{U}$-modules U_λ.

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If $\mu \in \Lambda$, define its *opposite* μ^0 to be $\tau_0(\mu + \delta) - \delta$, τ_0 the unique element of the Weyl group which interchanges positive and negative roots. Our main result can now be formulated.

THEOREM A. *Set* $\lambda = (\mu - \delta)^0$. *Then* U_λ *occurs precisely once as a* $\mathscr{U}$*-summand of* T_μ *and is stable under* $\mathbf{G}$, *therefore is also a projective* $\mathscr{R}$*-module involving* R_λ *as a summand. In particular,* $\dim R_\lambda \leqq \dim U_\lambda$.

The proof uses some ideas from [**4**]. For $\mathbf{G} = \mathbf{SL}(2, K)$, a result of this type was first noticed empirically by the second author.

REMARKS. (1) One can effectively compute (at least for small rank and small p) the modules T_μ, starting with the known decomposition of tensor products of irreducible modules in characteristic 0 and then reducing modulo p. Using this approach and other data, the first author computed the Cartan invariants of $\mathbf{SL}(3, 5)$, avoiding Brauer's method.

(2) From the tensor product construction (and knowledge of the modules M_μ) one also gets an effective, but lengthy, algorithm for computing the "decomposition" numbers $d_{\lambda\lambda}$ which figure in [**4**]. This in turn yields the Cartan invariants of $\mathscr{U}$.

Call $\lambda \in \Lambda$ *regular* if $\lambda = \sum m_i\lambda_i$ with all m_i nonzero ($\lambda_i \in \Lambda$ fundamental dominant weights). Empirical evidence, along with some heuristic arguments, suggests the following conjecture, which is true in rank 1 ([**5**], [**6**]) and also for $G = \mathbf{SL}(3, 5)$, $\mathbf{SL}(3, 3)$, $\mathbf{Spin}(5, 3)$.

CONJECTURE. As $\mathscr{R}$-modules, $U_\lambda = R_\lambda$ if and only if λ is regular.

For the groups G_q, one obtains (as in [**5**], [**6**]):

THEOREM B. *If* $\boldsymbol{\lambda} = \lambda_0 + \lambda_1 p + \cdots + \lambda_k p^k$, *define* $U_{\boldsymbol{\lambda}} = U_{\lambda_0} \otimes U_{\lambda_1}^{(p)} \otimes \cdots \otimes U_{\lambda_k}^{(p^k)}$ (*as module for* $\mathbf{G}$). *Then* $U_{\boldsymbol{\lambda}}$ *is a projective* $\mathscr{R}_q$*-module* ($q = p^{k+1}$), *with* $R_{\boldsymbol{\lambda}}$ *as a direct summand.*

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