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EXTREMAL PLANE QUASICONFORMAL MAPPINGS WITH GIVEN BOUNDARY VALUES

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1. **Introduction.** Let Ω_i, $i = 1, 2$, be regions in the complex plane, $f(z)$ a quasiconformal mapping of Ω_1 onto Ω_2. Let Q_f denote the class of all quasiconformal mappings of Ω_1 onto Ω_2 which have the same boundary values as f. A mapping $f^* \in Q_f$ will be called *extremal* for its boundary values if it is K^*-quasiconformal and if there exists no K-quasiconformal mapping in Q_f with $K < K^*$. The quantity $K^* = K^*(f)$ is the extremal dilatation for the class Q_f. (As is well known [**3**], there may be more than one K^*-quasiconformal mapping in the class Q_f.) In the present account, which is only an abstract, we restrict ourselves to the case $\Omega_1 = \Omega_2 = E = \{|z| < 1\}$. Generalizations to Riemann surfaces will be referred to in a detailed account, giving proofs, further results, and applications that is to appear elsewhere.

In what follows, the L^1 norm $\int\int_E |\varphi(z)|\,dx\,dy$ of functions $\varphi(z)$ *holomorphic* in E will be denoted by $\|\varphi\|$.

In 1969, R. S. Hamilton [**1**] proved the following: *If $f^* \in Q_f$ is an extremal mapping, $\kappa^*(z) = f^*_{\bar z}/f^*_z$, then*

$$\sup_{\|\varphi\| \leqq 1} \left| \iint_E \kappa^*(z)\varphi(z)\,dx\,dy \right| = k^*(f) = \frac{K^*(f) - 1}{K^*(f) + 1}. \tag{1.1}$$

A central result of the present work is (§3) that condition (1.1) *characterizes* extremal mappings of E.

2. **Estimates for** $K^*(f)$. The following is a generalization of an inequality proved in [**2**] from the case $K^*(f) = 1$ to arbitrary $K^*(f)$.

THEOREM 2.1. *If $f(z)$ is a quasiconformal self-mapping of E, $\kappa(z) = f_{\bar z}/f_z$, and if $\varphi(z)$ is holomorphic in E, then*

$$\left| \iint_E \frac{\kappa(z)}{1-|\kappa(z)|^2}\varphi(z)\,dx\,dy \right| \leqq \frac{k^*(f)}{1+k^*(f)}\|\varphi\| + \iint_E \frac{|\kappa(z)|^2}{1-|\kappa(z)|^2}|\varphi(z)|\,dx\,dy. \tag{2.1}$$

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[1] Supported by GP 27275.

For certain special holomorphic functions $\varphi(z)$ the quantity on the left side of (2.1) can be estimated in a useful way from below. Let us select n equally spaced points on $\{|z| = 1\}$, $n \geqq 4$. These are mapped by $w = f(z)$ onto n points of $\{|w| = 1\}$. The unique extremal quasiconformal mapping of E onto E, which preserves this correspondence of n-gons, is [**4**] a Teichmüller mapping with complex dilatation

$$k_n(\overline{\varphi_n(z)}/|\varphi_n(z)|),$$

where $\varphi_n(z)$ is holomorphic in E and has finite norm $\|\varphi_n\|$. Without loss of generality we can assume that $\|\varphi_n\| = 1$. It is then possible to prove the following:

THEOREM 2.2. *Let $f(z)$ be a quasiconformal self mapping of E, $\kappa(z) = f_{\bar z}/f_z$. Let $n \geqq 4$ points be selected on ∂E and k_n and $\varphi_n(z)$ determined as above. Then*

$$\left| \iint_E \frac{\kappa(z)}{1 - |\kappa(z)|^2} \varphi_n(z)\, dx\, dy \right| \geqq \frac{k_n}{1 - k_n} - \iint_E \frac{|\kappa(z)|^2}{1 - |\kappa(z)|^2} |\varphi_n(z)|\, dx\, dy.$$

Given f, let us introduce the quantities

$$I(f) = \sup_{\|\varphi\| \leqq 1} \left| \iint_E \frac{\kappa(z)\varphi(z)}{1 - |\kappa(z)|^2}\, dx\, dy \right| \quad (\kappa(z) = f_{\bar z}/f_z),$$

$$\Delta(f) = \sup_{\|\varphi\| \leqq 1} \iint_E \frac{|\kappa(z)|^2 |\varphi(z)|}{1 - |\kappa(z)|^2}\, dx\, dy.$$

The following then follows easily as a corollary of Theorems 2.1 and 2.2:

THEOREM 2.3. *The maximal dilatation $K^*(f)$ which is extremal for the class Q_f satisfies the estimate*

$$\frac{1}{1 - 2I(f) + 2\Delta(f)} \leqq K^*(f) \leqq 1 + 2I(f) + 2\Delta(f).$$

REMARK. When f is a Teichmüller mapping with finite norm both inequalities become equalities.

3. **Characterization of complex dilatations of extremal mappings.** Let

$$H(f) = \sup_{\|\varphi\| \leqq 1} \left| \iint_E \kappa(z)\varphi(z)\, dx\, dy \right| \cdot$$

By means of Theorem 2.2 one obtains a new proof of Hamilton's necessary condition (1.1), with an explicit identification of an extremal sequence for $H(f)$:

THEOREM 3.1. *If $f^* \in Q_f$ is extremal, $\kappa^*(z) = f^*_{\bar z}/f^*_z$, then*

$$\lim_{n\to\infty}\left|\iint_E \kappa^*(z)\varphi_n(z)\,dx\,dy\right| = H(f) = k^*(f).$$

Furthermore, as a deduction from Theorem 2.3, the converse also follows:

THEOREM 3.2. *A necessary and sufficient condition for the quasiconformal mapping f of E onto E to be extremal for its boundary values is that*

$$H(f) = \operatorname*{ess\,sup}_{z\in E} |\kappa(z)|.$$

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