

BOUNDED HARMONIC BUT NO DIRICHLET-FINITE HARMONIC

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ABSTRACT. The purpose of the present note is to announce that for each $n \geq 3$ there exists a Riemannian n -manifold, which carries nonconstant bounded harmonic functions but no nonconstant Dirichlet-finite harmonic functions.

1. A C^2 -function u on a Riemannian n -manifold M is harmonic on M if $\Delta u = 0$, where

$$\Delta u = \frac{-1}{g^{1/2}} \sum_{i,j=1}^n \frac{\partial}{\partial x^i} \left(g^{1/2} g^{ij} \frac{\partial u}{\partial x^j} \right).$$

Here (g_{ij}) is the metric tensor for M , $(g^{ij}) = (g_{ij})^{-1}$, and $g = \det(g_{ij})$.

It is not known (cf. Sario-Nakai [4, p. 406]) whether or not for each $n \geq 3$ there exists a Riemannian n -manifold M which carries nonconstant bounded harmonic functions but every harmonic function u on M is a constant whenever its Dirichlet integral

$$D(u) = \int_M \sum_{i,j=1}^n g^{ij} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j} dV < \infty,$$

where $dV = g^{1/2} dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$ is the volume element. For $n = 2$ the problem was solved in the affirmative by Tôki [5], his example known as Tôki's example. Royden [2] and Sario [3] also obtained a similar result.

The purpose of the present note is to announce that for each $n \geq 3$ there does exist a Riemannian n -manifold which solves the problem in the affirmative.¹ Details will be published elsewhere.

2. Fix $n \geq 3$. Denote by M_0 the punctured Euclidean n -space $R^n - 0$ with the metric tensor

$$g_{ij}(x) = |x|^{-4} (1 + |x|^{n-2})^{4/(n-2)} \delta_{ij}, \quad 1 \leq i, j \leq n,$$

where $|x| = [\sum_{i=1}^n (x^i)^2]^{1/2}$ for $x = (x^1, x^2, \dots, x^n) \in M_0$.

LEMMA. *Every positive harmonic function u on M_0 has the form:*

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¹ Professor Sario informed me that he recently obtained a similar result with Professors Wang and Hada.

$$u(x) = a/(1 + |x|^{n-2}) + b$$

for some constants a, b .

3. For each pair (m, l) of positive integers m, l , and $k = z^{m-1}(2l - 1) - 1$, set

$$H_{ml} = \{8^k x = (8^k x^1, 8^k x^2, \dots, 8^k x^n) \in M_0 \mid |x| = 1 \text{ and } x^1 \geq 0\},$$

$$H'_{ml} = \{8^{-k} x = (8^{-k} x^1, 8^{-k} x^2, \dots, 8^{-k} x^n) \in M_0 \mid |x| = 1 \text{ and } x^1 \geq 0\}.$$

Denote by M'_0 the manifold obtained from M_0 by deleting all the closed hemispheres H_{ml} and H'_{ml} .

Take two sequences $\{M'_0(l)\}_1^\infty$ and $\{M''_0(l)\}_1^\infty$ of duplicates of M'_0 . For each fixed $m \geq 1$ and subsequently fixed $j \geq 0$ and $1 \leq i \leq m$, connect $M'_0(i + mj)$ with $M''_0(i + m + mj)$ for even j and $M'_0(i + mj)$ with $M''_0(i - m + mj)$ for odd j , crosswise along all the hemispheres H_{ml} and H'_{ml} ($l \geq 1$). The resulting Riemannian n -manifold N is an infinitely sheeted covering manifold of M_0 .

THEOREM. For the manifold N the following are true:

$$(1) \dim HB(N) = 2, \quad (2) \dim HD(N) = 1,$$

where H is the space of harmonic functions, D the space of Dirichlet-finite functions, B the space of bounded functions and HX stands for $H \cap X$.

It can be shown that every bounded harmonic function u on N takes the same value at all the points in N which lie over the same point in M_0 . Therefore the function u is Dirichlet-finite only if u is a constant. Here we use the q -Lemma (cf. Rodin-Sario [1, p. 39]).

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