

PRIMES WHICH ARE REGULAR FOR ASSOCIATIVE H -SPACES

BY JOHN R. UNDERWOOD

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This note summarizes some results on associative H -spaces with the homotopy type of a finite CW-complex. Such spaces are called *finite-dimensional associative H -spaces*. The main theorem generalizes a result of Serre [5] for the simple classical groups.

A well-known theorem of Hopf [2] states that the rational cohomology $H^*(X, \mathbf{Q})$ of a finite-dimensional H -space is an exterior algebra $\Lambda(X_{2N_1-1}, \dots, X_{2N_K-1})$, where the dimension of X_{2N_i-1} is $2N_i - 1$ and $N_1 \leq \dots \leq N_K$. With this in mind, suppose that X is a finite-dimensional associative H -space with

$$H^*(X, \mathbf{Q}) = \Lambda(X_{2N_1-1}, \dots, X_{2N_K-1}), \quad \text{where } N_1 \leq \dots \leq N_K.$$

Form the space

$$Y = S^{2N_1-1} \times \dots \times S^{2N_K-1}.$$

One says that a prime p is *regular* for X if there is a function $f: Y \rightarrow X$ which induces an isomorphism in cohomology with $\mathbf{Z}/p\mathbf{Z}$ coefficients. The following theorem is due to Serre [5].

THEOREM. *If X is a simply connected, simple, compact, connected classical group with rational cohomology as above, then p is regular for X if and only if $p \geq N_K$.*

Such groups are of the form $SU(N)$, $Sp(N)$ or $Spin(N)$. Serre's proof is a case by case study of these groups. Various generalizations of this theorem have appeared in the literature. Kumpel [3], for example, verified that the conclusion of Serre's theorem is true for the exceptional Lie groups, G_2 , F_4 , E_6 , E_7 and E_8 . I am announcing a result which generalizes Serre's to finite-dimensional associative H -spaces. Needless to say, certain mild restrictions are necessary. For example, certain assumptions about primitive generation are necessary.

Suppose that X is an H -space with multiplication

$$X \times X \xrightarrow{\mu} X.$$

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Let p_1 and p_2 be the projections

$$X \times X \begin{matrix} \xrightarrow{p_1} \\ \xrightarrow{p_2} \end{matrix} X$$

onto the first and second factors, respectively. One says that an element $x \in H^*(X, R)$ is *primitive* if

$$\mu^*(x) = p_1^*(x) + p_2^*(x).$$

$H^*(X, R)$ is *primitively generated* if it is generated by primitives as an R -algebra.

I can now state the main theorem.

THEOREM I. *Let X be a simply connected associative H -space of finite dimension with*

$$H^*(X, \mathbf{Q}) = \Lambda(X_{2N_1-1}, \dots, X_{2N_K-1}), \quad \text{where } N_1 \leq \dots \leq N_K.$$

Let p be a prime. Then p is regular for X if and only if

- (1) $p \geq N_K$,
- (2) $H^*(X, \mathbf{Z}/p\mathbf{Z})$ is primitively generated,
- (3) $H^*(X, \mathbf{Z})$ has no p -torsion.

Larry Smith [6] has shown that (1)–(3) are sufficient for p to be a regular prime; so the only remaining point is the necessity of the conditions. Conditions (2) and (3) are immediate and (1) is handled by the following proposition.

PROPOSITION. *With notation as above, if $p < N_K$, p a prime, then p is not regular for X .*

To prove this, one constructs a space $X_{(p)}$ called the localization of X at the prime p (see Sullivan [9] for details). One needs the facts that $H^*(X_{(p)}, \mathbf{Z}) \approx H^*(X, \mathbf{Z})_{(p)}$ where the latter denotes the group-theoretic localization (Atiyah and Macdonald [1]). If X is an associative H -space then so is $X_{(p)}$ (Mislin [4]); and finally, if, given the above notation, p is regular for X , then $X_{(p)} \simeq Y_{(p)}$. The Proposition now follows from

THEOREM II. *If $Y \simeq S^{2N_1-1} \times \dots \times S^{2N_K-1}$, where $N_1 \leq \dots \leq N_K$ and p is an odd prime less than N_K , then $Y_{(p)}$ is not an associative H -space.*

The proof, which employs Stasheff’s notion of an A_p -structure [7], is essentially an Adam’s operation computation.

The following proposition is a scholium.

PROPOSITION. *Let $\alpha \in \pi_7(BSU(3))$ classify the principal fibration*

$SU(3) \rightarrow SU(4) \rightarrow S^7$, let E_3 be the induced fibration in

$$\begin{array}{ccccc}
 SU(3) & = & SU(3) & = & SU(3) \\
 \downarrow & & \downarrow & & \downarrow \\
 E_3 & \longrightarrow & SU(4) & \longrightarrow & * \\
 \downarrow & & \downarrow & & \downarrow \\
 S^7 & \xrightarrow{\beta} & S^7 & \xrightarrow{\alpha} & B(SU(3))
 \end{array}$$

then E_3 is not a homotopy associative H -space.

This answers Question 17 in a list of questions presented to the Neuchatel Conference on H -spaces, August 1970 [8].

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DEPARTMENT OF MATHEMATICS, RICE UNIVERSITY, HOUSTON, TEXAS 77001