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NONUNIFORMLY ELLIPTIC EQUATIONS: POSITIVITY OF WEAK SOLUTIONS

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1. This note is concerned with the weak boundary value problem

$$(1)\quad \int_\Omega \left(\sum_{i,j=1}^N a_{ij}(x) u_{x_i} v_{x_i} + b(x)uv \right) dx = \int_\Omega c(x) f v \, dx, \quad \text{all } v \in C_0^\infty(\Omega),$$

and the weak eigenvalue problem

$$(2)\quad \int_\Omega \left(\sum_{i,j=1}^N a_{ij}(x) u_{x_i} v_{x_i} + b(x)uv \right) dx = \lambda \int_\Omega c(x) uv \, dx, \quad \text{all } v \in C_0^\infty(\Omega),$$

where Ω is a connected open set in R^N. Our hypothesis concerning the coefficient matrix (a_{ij}) in (1) and (2) is similar to but weaker than those imposed on the elliptic operators which are studied in [**2**], [**3**], [**4**]. Specifically, we assume that $A = (a_{ij})$ is a real matrix-valued function, symmetric and positive definite almost everywhere on Ω with

$$(3)\qquad \|A\|, \|A^{-1}\| \in L^1_{\mathrm{loc}}(\Omega).$$

Concerning the coefficients b, c our assumptions are the following: b and c are real valued,

$$(4)\qquad Mb \geqq c > 0 \quad \text{a.e. on } \Omega$$

for some positive constant M and

$$(5)\qquad b, b^{-1}, c \in L^1_{\mathrm{loc}}(\Omega).$$

Under these assumptions we prove: If $f \in L^2(\Omega, c(x)\,dx)$, $f(x) \geqq 0$ a.e. on Ω and $f \neq 0$ then (1) has a solution positive almost everywhere on Ω, in particular a nonnegative eigenfunction of (2) is positive almost everywhere in Ω; if (2) has a nonnegative eigenfunction corresponding to an eigenvalue $\lambda_1 > 0$ then λ_1 is simple and the spectrum of (2) is contained in the interval $[\lambda_1, \infty]$.

This research was motivated by certain problems arising in connection with the study in [**1**] of nonlinear elliptic eigenvalue problems.

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2. We assume that (a_{ij}), b and c are as above. Let $\mathring{X}_1$ denote the set of functions $u \in C^\infty(\Omega)$ for which

$$\|u\|_{X_1}^2 = \int_\Omega \left(\sum_{i,j=1}^N a_{ij} u_{x_i} u_{x_j} + b u^2 \right) dx < \infty, \tag{6}$$

and let X_1 be the Hilbert space obtained by completing $\mathring{X}_1$ in the norm (6); X_0 will denote the closure of $C_0^\infty(\Omega)$ in X_1.

LEMMA 1. *The space X_1 is stronger than the space $H_{\mathrm{loc}}^{1,1}(\Omega)$, and the norm on X_1 is given by* (6). *The spaces X_1 and X_0 are closed under the operation*

$$u \to |u|; \tag{7}$$

moreover, this operation is norm preserving in X_1.

Here, as is standard, $H_{\mathrm{loc}}^{1,1}(\Omega)$ denotes the Fréchet space of locally integrable, locally strongly L^1 differentiable functions on Ω.

Let X be any closed subspace of X_1 with

$$X_0 \subset X \subset X_1, \tag{8}$$

and such that X is closed under the mapping (7), and let Y denote the Hilbert space consisting of measurable functions f for which

$$\|f\|_y^2 = \int_\Omega |f|^2 c(x)\, dx < \infty.$$

From (4) and (6) it is clear that the functions in X are also in Y. The inclusion mapping $X \subset Y$ will be denoted by i.

LEMMA 2. *The mapping $i: X \to Y$ is bounded, injective, and has dense range. The mapping $i^*: Y \to X$ (the Lax-Milgram operator) is also injective with dense range and preserves nonnegativity.*

It is not difficult to see that when $X = X_0$ then $u = i^*f$, for $f \in Y$, is the solution of (1).

We next consider the "Green's operator" $k = ii^*$ in Y, and state the first of our two main results which refines the nonnegativity assertion of Lemma 2.

THEOREM 1. *The operator k is selfadjoint, positive definite, and bounded. If f is a nonzero element of Y and $f(x) \geqq 0$ a.e. on Ω then $h = kf$ satisfies*

$$h(x) > 0 \quad \textit{a.e. on } \Omega. \tag{9}$$

In particular if k has a nonnegative eigenfunction φ, then

$$\varphi(x) > 0 \quad \textit{a.e. on } \Omega.$$

REMARK. If Ω is bounded, $b = 0$, and the coefficients in (2) satisfy stronger regularity conditions then such a positivity result can be obtained from Lemma 2 and the Harnack inequality of Trudinger [**4**]; indeed in that case one can assert, instead of merely (9), that h has a positive essential lower bound on each compact subset of Ω. Our proof of Theorem 1 however makes use of global rather than local methods.

THEOREM 2. *Let φ be a nonnegative eigenfunction of k, $\mu\varphi = k\varphi$, then $\|k\| = \mu$, and μ is a simple eigenvalue of k.*

While Theorem 2 is very easily proved in the case where k is compact, the general case is somewhat deeper and does not seem to be contained in the extensive literature on positive operators.

3. We now describe the sort of application of Theorems 1 and 2 which was wanted for [**1**]. We consider the problem (2) in $W_0^{1,p}(\Omega)$ for some p with $2 \leqq p \leqq \infty$. With p fixed we take

$$r = p/(p-2)$$

and we take s to be an element of the extended real number system with $p \leqq s$ and $s \leqq Np/(N-p)$, if $p \leqq N$, finally we take

$$r_1 = s/(s-2).$$

We assume that the matrix A is as in §1 and in addition that

$$\|A\| \in L^r(\Omega).$$

Concerning b and c we assume only that

$$b, c \in L^{r_1}(\Omega).$$

THEOREM 3. *Let $u \in W_0^{1,p}(\Omega)$ be a nonnegative eigenfunction of the weak problem* (2) *corresponding to the eigenvalue $\lambda_1 > 0$. Then*

$$u(x) > 0 \quad a.e.\ in\ \Omega,$$

and, for all $v \in W_0^{1,p}(\Omega)$,

$$\int_\Omega \left(\sum_{i,j=1}^N a_{ij} v_{x_i} v_{x_j} + bv^2 \right) dx \geqq \lambda_1 \int_\Omega v^2 c(x)\, dx,$$

with equality only if v is proportional to u.

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