chains. An *addition chain for n* is a sequence

$$1 = a_0, a_1, \ldots, a_r = n$$

such that

$$a_i = a_j + a_k \quad \text{for some } k \leqq j < i.$$

Most work on addition chains is concerned with the calculation of $l(n)$, the length of the shortest addition chain for n. The relationship of these chains to computer algorithms is that an $l(n)$-chain gives a way to form x^n, arbitrary x and integral n, with fewest multiplications. Of special interest to mathematicians is the still unanswered "Scholz-Brauer conjecture"

$$l(2^n - 1) \leqq n - 1 + l(n).$$

Knuth's paper consolidates relevant mathematical and empirical work on addition chains; it is certainly of more interest to the mathematician than to the computer scientist. Recent computer work by Knuth (appearing in the second printing) has turned up the remarkable fact that $l(12509) < l^*(12509)$—an l^*-chain is one in which $j = i - 1$.

4. For various reasons, many of them nontechnical and extraneous, the interaction between mathematics and computer science over the past two decades has been less than one might expect. This has been and is unfortunate as each field can significantly benefit from and contribute to the other. The real importance of Knuth's work is that it represents a truly positive step towards eliminating the existing breach between mathematicians and computer scientists.

MARK B. WELLS

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Structure and Representations of Jordan Algebras, by Nathan Jacobson. American Mathematical Society Colloquium Publications, vol. 39. American Mathematical Society, Providence, R. I., 1968. x + 453 pp. $11.30 (Member Price $8.48).

In 1932 the physicist P. Jordan, with a view toward generalization of the quantum mechanical formalism, initiated the study of the class of nonassociative algebras which now bear his name and are the subject of this book. A Jordan algebra $\mathfrak{J}$ over a field Φ of characteristic $\neq 2$ is a commutative (nonassociative) algebra over Φ satisfying the Jordan identity

$$(a^{\cdot 2}.b).a = a^{\cdot 2}.(b.a),$$

where we have written $a.b$ for the product in $\mathfrak{J}$ and $a^{\cdot 2} = a.a$. Rapid development of Jordan's ideas culminated in 1934 in the classification by Jordan, J. von Neumann and E. Wigner of those finite-dimensional real Jordan algebras which are formally real in the sense that $a_1^{\cdot 2} + a_2^{\cdot 2} + \cdots + a_k^{\cdot 2} = 0$ implies $a_1 = a_2 = \cdots = a_k = 0$. There was little further activity for a dozen years until A. A. Albert, as part of his general study of nonassociative algebras, began to develop a structure theory for finite-dimensional Jordan algebras over Φ. This was modelled on the known theories for Lie algebras of characteristic 0 and associative algebras: definition of the radical, characterization of semisimple algebras as direct sums of simple ideals, and classification of simple algebras. It is characteristic of the point of view of this book that these early results appear only after 200 pages as corollaries of more general theorems.

In 1948 the author of this book turned his attention to Jordan rings and algebras. Much of the further development of the subject is due to him, to his students, and to others who have been strongly influenced by him. This book not only incorporates a major portion of the author's research over a twenty-year period, but also is dominated by the fundamental contributions he has made. The book is very carefully organized, and a study of it is indispensable for anyone seriously interested in Jordan algebras.

There are nine chapters and a section entitled "Further results and open questions". Each chapter is crowded with new concepts and important theorems. This results in a very elaborate machinery which the author employs consistently throughout the book. We can give here only a very superficial listing of some of the contents of each chapter.

The first chapter is concerned with "Foundations". If, in an associative algebra $\mathfrak{A}$ over Φ, the associative product ab is replaced by the Jordan product

$$a.b = \tfrac{1}{2}(ab + ba),$$

one obtains a Jordan algebra $\mathfrak{A}^+$ from $\mathfrak{A}$. A Jordan algebra $\mathfrak{J}$ over Φ is called special if there exists a monomorphism of $\mathfrak{J}$ into an algebra $\mathfrak{A}^+$, $\mathfrak{A}$ associative; otherwise $\mathfrak{J}$ is called exceptional. In this chapter free special Jordan algebras and free Jordan algebras are studied. Also the important Jordan triple product

$$\{abc\} = (a.b).c + (b.c).a - (a.c).b$$

is introduced. Basic linear operators are the right multiplication R_a and

$$U_a = 2R_{a^2} - R_{a^{\cdot 2}}: \quad x \to \{axa\}.$$

The author proves Macdonald's theorem, which implies that any three variable identity which is of degree at most one in one of the variables and which holds for all special Jordan algebras is valid for all Jordan algebras. As a consequence one has the fundamental identity

$$U_a U_b U_a = U_{\{aba\}}$$

in any Jordan algebra. The first chapter concludes with a discussion of invertible elements, homotopy, isotopy, and the structure group.

The second chapter is devoted to "Elements of representation theory". Associative specializations and multiplication specializations of Jordan algebras $\mathfrak{J}$ are introduced. An associative specialization of $\mathfrak{J}$ in an associative algebra $\mathfrak{G}$ with 1 is a homomorphism of $\mathfrak{J}$ into the special Jordan algebra $\mathfrak{G}^+$, while a multiplication specialization ρ of $\mathfrak{J}$ in $\mathfrak{G}$ is a linear mapping ρ of $\mathfrak{J}$ into $\mathfrak{G}$ such that

$$[a^\rho, (a^{\cdot 2})^\rho] = 0,$$

$$2a^\rho b^\rho a^\rho + (a^{\cdot 2}.b)^\rho = 2a^\rho (a.b)^\rho + b^\rho (a^{\cdot 2})^\rho.$$

The concept of a Jordan bimodule $\mathfrak{M}$ for $\mathfrak{J}$ is equivalent to that of a multiplication specialization of $\mathfrak{J}$ in $\mathrm{Hom}_\Phi(\mathfrak{M}, \mathfrak{M})$. The special universal envelope $S(\mathfrak{J})$ and the universal multiplication envelope $U(\mathfrak{J})$ are studied; correspondingly, the unital envelopes $S_1(\mathfrak{J})$ and $U_1(\mathfrak{J})$ for $\mathfrak{J}$ with 1. The average of any two commuting associative specializations is a multiplication specialization. This leads to the notion of a special Jordan bimodule and also to that of the squared special universal envelope of $\mathfrak{J}$. If $\mathfrak{J}$ is a Jordan algebra with 1, then $U(\mathfrak{J}) = \Phi z \oplus S_1(\mathfrak{J}) \oplus U_1(\mathfrak{J})$. Examples are given which are used in later chapters.

The third chapter is called "Peirce decompositions and Jordan matrix algebras". The basic properties of the Peirce decomposition of a Jordan algebra relative to an idempotent, or relative to a finite set of orthogonal idempotents whose sum is 1, are derived. If $\mathfrak{D}$ is a nonassociative algebra with 1 and involution j ($d \to \bar{d}$), and if a is a diagonal matrix

$$\mathrm{diag}\{a_1, \ldots, a_n\}$$

whose diagonal entries a_i are j-symmetric elements of the nucleus of $\mathfrak{D}$ with inverses in the nucleus, then

$$J_a : X \to a^{-1}\bar{X}^t a$$

is an involution of $\mathfrak{D}_n$, and the set $\mathfrak{H}(\mathfrak{D}_n, J_a)$ of J_a-symmetric elements of $\mathfrak{D}_n$ is a commutative algebra relative to the multiplication

$$A.B = \tfrac{1}{2}(AB + BA).$$

If $\mathfrak{H}(\mathfrak{D}_n, J_a)$ is a Jordan algebra, it is called a Jordan matrix algebra; if $n \geqq 3$, this is the case if and only if either $\mathfrak{D}$ is associative or $n = 3$ and $\mathfrak{D}$ is alternative with symmetric elements in the nucleus. The following Coordination Theorem is achieved: If $\mathfrak{J}$ is a Jordan algebra with 1 which is a sum of $n \geqq 3$ connected nonzero orthogonal idempotents, then $\mathfrak{J}$ is isomorphic to a Jordan matrix algebra. Some basic results on representation theory of Jordan matrix algebras are obtained.

The fourth chapter is concerned with "Jordan algebras with minimum condition on quadratic ideals". Here the author gives his analogue for Jordan algebras of the associative theory of artinian rings. The basic concept is that of a quadratic ideal $\mathfrak{B}$ of $\mathfrak{J}$: a subspace $\mathfrak{B}$ of $\mathfrak{J}$ satisfying

$$\mathfrak{J}U_b \subseteq \mathfrak{B} \quad \text{for every } b \text{ in } \mathfrak{B}.$$

The author studies Jordan algebras $\mathfrak{J}$ with 1 such that $\mathfrak{J}$ is nondegenerate (that is, $U_a = 0$ in $\mathfrak{J}$ implies $a = 0$) and such that two minimum conditions on quadratic ideals in $\mathfrak{J}$ are satisfied. Any such Jordan algebra is a direct sum of a finite number of ideals which are simple algebras satisfying the same conditions. Simple algebras satisfying the conditions are classified in terms of Jordan division algebras, the Jordan algebra of a nondegenerate symmetric bilinear form, and certain Jordan matrix algebras.

The fifth chapter is called "Structure theory for finite-dimensional Jordan algebras". Here the theory of solvable ideals is developed, and the radical of a finite-dimensional Jordan algebra $\mathfrak{J}$ is defined to be its maximal solvable ideal. Since a finite-dimensional $\mathfrak{J}$ is nondegenerate if and only if it is semisimple, Albert's structure theory for semisimple algebras is obtained from the results of the previous chapter. Representation theory is employed to classify the finite-dimensional special central simple Jordan algebras of degree $n \geqq 3$ in terms of finite-dimensional central simple associative algebras with involution. The only finite-dimensional exceptional central simple Jordan algebras are of degree 3 and dimension 27; these are studied in detail in the final chapter.

The sixth chapter is entitled "Generic minimum polynomials, traces and norms". The differential calculus of rational mappings of finite-dimensional vector spaces is developed, and is used in the study of the generic minimum polynomial of any finite-dimensional strictly power-associative algebra with 1. Examples are given, listing the generic minimum polynomials for all finite-dimensional central simple Jordan algebras. The generic norm n satisfies $n(\{aba\}) = n(a)^2 n(b)$. A finite-dimensional

Jordan algebra with 1 is separable (that is, remains semisimple under any scalar extension) if and only if its generic trace form t is nondegenerate.

In the seventh chapter the author studies "Representation theory for separable Jordan algebras". One of the main results here is that a finite-dimensional Jordan algebra $\mathfrak{J}$ with 1 is separable if and only if the associative algebra $U(\mathfrak{J}) = \Phi z \oplus S_1(\mathfrak{J}) \oplus U_1(\mathfrak{J})$ is separable. This is proved by use of the classification of central simple Jordan algebras, and computation of $S_1(\mathfrak{J})$ and $U_1(\mathfrak{J})$ for central simple algebras $\mathfrak{J}$. A detailed discussion of Clifford algebras and meson algebras is included for the degree 2 case. Bimodules for composition algebras are employed for the cases of degree $\geqq 3$. The chapter concludes with the radical splitting theorem, and the uniqueness of this decomposition for characteristic 0.

The eighth chapter is called "Connections with Lie algebras". Here a number of results of the preceding chapter are proved (for characteristic 0) by Lie algebra methods without using the classification of central simple Jordan algebras. A basic notion is the associator Lie triple system of a Jordan algebra $\mathfrak{J}$, which arises from the formula

$$[[R_c R_a] R_b] = R_{[a,b,c]}$$

where $[a, b, c]$ is the associator $[a, b, c] = (a.b).c - a.(b.c)$ in $\mathfrak{J}$. The author presents the Tits-Koecher construction of a Lie algebra from a Jordan algebra. He concludes the chapter with his theory of Cartan subalgebras of a finite-dimensional Jordan algebra, and applies it to give a proof (without classification theory) that the generic trace form of a finite-dimensional separable Jordan algebra is nondegenerate.

The final chapter is devoted to "Exceptional Jordan algebras", specifically to the finite-dimensional exceptional central simple algebras. These are the (27-dimensional) algebras $\mathfrak{H}(\mathfrak{O}_3, J_\gamma)$, where $\mathfrak{O}$ is an algebra of octonions (that is, a Cayley algebra), and the Jordan division algebras which become these upon extension of the base field. It is proved that $\mathfrak{H}(\mathfrak{O}_3, J_\gamma)$ and $\mathfrak{H}(\mathfrak{O}'_3, J_{\gamma'})$ are isomorphic if and only if the coefficient algebras $(\mathfrak{O}, j)$ and $(\mathfrak{O}', j)$ are isomorphic and the quadratic forms $Q(x) = \frac{1}{2}t(x^{\cdot 2})$ defined by the two algebras are equivalent. Moufang projective planes are studied via the algebras $\mathfrak{H}(\mathfrak{O}_3, J_\gamma)$. The connections between exceptional central simple Jordan algebras and exceptional simple Lie algebras are treated briefly (and more fully in the author's recent book, *Exceptional Lie Algebras*). The discussion of exceptional Jordan division algebras is based on a construction by J. Tits.

As indicated before, the final section of the book is devoted to further results and open questions. Perhaps the most important of these is the theory of quadratic Jordan algebras of K. McCrimmon, which extends the subject of this book to algebras over an arbitrary commutative ring with 1 (including characteristic 2). There are many exercises throughout the book, a bibliography and an index. Also there are a number of trivial misprints, none of which should cause the reader difficulty.

What this sketchy listing of some of the contents of this book fails to convey is the superb organization of the vast amount of material which the author has included. Representation theory is basic to the entire presentation. It is not only that individual topics are interesting in themselves as they are developed, but their impact is felt consistently throughout the remaining pages. This is a book which one can come back to again and again, gaining new insights every time.

R. D. SCHAFER

Introduction to the Arithmetic Theory of Automorphic Functions by Goro Shimura. Iwanami Shoten, Publishers, Tokyo; Princeton University Press, Princeton, New Jersey.

Shimura's book *Introduction to the Arithmetic Theory of Automorphic Functions* is one of the most significant additions to the literature on automorphic function theory in many years. In this remarkably slender volume, Professor Shimura manages to build up the theory of automorphic functions from scratch and at the same time present much recent work which was previously available only in the original papers.

Chapter 1 is devoted to a study of a discrete subgroup Γ of $SL(2, \boldsymbol{R})$ acting on the complex upper half-plane H by linear fractional transformations. If H^* denotes the union of H and the cusps of Γ, then it is shown that the quotient space H^*/Γ carries a natural structure of a Riemann surface. If the Riemann surface H^*/Γ is compact, then Γ is said to be a *Fuchsian group of the first kind*. The theory is illustrated by means of the elliptic modular group and its congruence subgroups, all Fuchsian groups of the first kind, and the genera of all these Riemann surfaces are computed.

Chapter 2 proceeds to study automorphic functions and automorphic forms with respect to a Fuchsian group of the first kind. The basic facts concerning compact Riemann surfaces are assumed. From these, the dimension of the spaces of holomorphic automorphic forms and cusp forms of given weight with respect to Γ are computed. Moreover, the volume of a fundamental domain for Γ is computed.

The arithmetic theory begins in Chapter 3 which is devoted to the theory of Hecke operators. Shimura develops the theory of Hecke operators axiomatically. At first, the Hecke operators appear as elements of an abstract ring, and only later do the classical Hecke operators arise from representations of the abstract ring on the spaces of automorphic forms of