

A CHARACTERIZATION OF INNER PRODUCT SPACES

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Let X be a real normed linear space and f a functional on X . Recall that by the first one-sided variation of f at x in the direction h we mean

$$f'_+(x)(h) = \lim_{t \rightarrow 0^+} \frac{[f(x + th) - f(x)]}{t}$$

and by the second one-sided variation of f at x in the directions h_1 and h_2 we mean

$$f''_+(x)(h_1, h_2) = \lim_{t \rightarrow 0^+} \frac{[f'_+(x + th_1)(h_2) - f'_+(x)(h_2)]}{t}.$$

Let

$$f(x) = \frac{1}{2}\|x\|^2 \quad \text{and} \quad (x, y) = f'_+(x)(y).$$

PROPOSITION 1. *Every normed linear space resembles an inner product space in the sense that*

- (i) (x, y) is well defined;
- (ii) $(x, x) \geq 0$ with equality if and only if $x = 0$;
- (iii) $\|x\| = (x, x)^{1/2}$.

Moreover, if X is an inner product space with inner product $[\cdot, \cdot]$, then $(\cdot, \cdot) = [\cdot, \cdot]$.

PROPOSITION 2. *The following are equivalent:*

- (i) X is an inner product space;
- (ii) (x, y) is symmetric;
- (iii) (x, y) is linear in the first variable;
- (iv) $f''_+(0)(x, y)$ is linear in the first variable.

The proofs of these and related results will appear elsewhere.

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