BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 3, May 1973

# TOPOLOGIES ON SPACES OF BAIRE MEASURES

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Communicated by R. Creighton Buck, November 6, 1972

**Introduction.** Let $X$ be a completely regular, Hausdorff space, let $C$ be the space of real-valued continuous functions on $X$ and let $C^b$ be the subspace of $C$ consisting of the uniformly bounded continuous functions on $X$. The Banach dual of $C^b$ (for the uniform norm) will be denoted by $M$, and the subspace of $M$ consisting of all totally-finite, signed Baire measures on $X$ will be denoted by $M_\sigma$. (Recall that the algebra of Baire sets is the smallest $\sigma$-algebra of subsets of $X$ for which each of the functions in $C$ is measurable.) Finally, the space of signed measures in $M_\sigma$ which have finite support will be denoted by $L$. By identifying each point of $X$ with the Dirac measure at that point we may assume that $X$ is a subset of $L$ (and hence of $M_\sigma$). The purpose of the present note is to describe some results recently obtained by the author concerning completions of $L$ relative to certain natural locally convex topologies on $L$, and some applications of these results. (For the proofs and for further details, the reader is referred to [**5**] and [**6**].) The principal results are essentially generalizations to arbitrary spaces of the following theorem due to M. Katětov and V. Pták. (See [**3**], [**4**] and [**8**].)

THEOREM. *Let $X$ be pseudocompact. Then the completion of $L$ for the topology of uniform convergence on the pointwise bounded, equicontinuous subsets of $C$ is the space $M_\sigma$ $(=M)$.*

In order to avoid certain technical difficulties in the discussion, we will assume throughout the paper that $X$ has a nonmeasurable cardinal. (As is well known, it is consistent with the axioms of set theory to assume that there are no measurable cardinals.) For a discussion of the results in the presence of measurable cardinals, the reader is referred to [**5**] and [**6**].

1. **The topology $e^b$.** A set $B \subset C$ is *equicontinuous* if for all $x \in X$ and for every positive number $\varepsilon$, there is a neighborhood $U$ of $x$ such that $|f(x) - f(y)| \leqq \varepsilon$ for all $y \in U$ and all $f \in B$. The set $B$ is *uniformly bounded* if there is a number $K$ such that $|f(x)| \leqq K$ for all $f \in B$ and all $x \in X$. Let $\mathscr{E}^b$ denote the family of all uniformly bounded, equicontinuous subsets of $C^b$; and let $e^b$ denote the topology on $M_\sigma$ of uniform convergence on the sets in $\mathscr{E}^b$. It is easily verified that $e^b$ is a locally convex topology on $M_\sigma$. We then have the following result:

*AMS (MOS) subject classifications* (1970). Primary 46A30, 60B10; Secondary 60B05.
*Key words and phrases*. Baire measures, weak convergence of Baire measures, complete spaces of Baire measures.

THEOREM 1.1. *Let $X$ be a completely regular Hausdorff space, and let $M_\sigma$ be endowed with the topology $e^b$. Then the following hold*:

1. *$M_\sigma$ is complete.*
2. *$L$ is dense in $M_\sigma$.*
3. *The dual space of $M_\sigma$ is $C^b$.*
4. *$X$ is a bounded subset of $M_\sigma$.*
5. *The restriction of the topology $e^b$ to $X$ is the original topology on $X$.*

The proof of the theorem can be outlined as follows. One first considers the space $L$ with the topology $e^b$. It is not difficult to show that the dual of $(L, e^b)$ is $C^b$, and that, in fact, $e^b$ is the finest locally convex topology on $L$ whose restriction to $X$ is the original topology and for which $X$ is a bounded subset. One then shows that the completion of $(L, e^b)$ is $(M_\sigma, e^b)$ from which the theorem follows easily. This is done by combining the well-known theorem of A. Grothendieck characterizing complete locally convex topologies [**7**, p. 270] with certain results from the theory of measures in topological spaces.

As an application of Theorem 1.1, the following result of E. Granirer [**2**] may be obtained. (Of course, $M_\sigma^+$ denotes the space of all finite Baire measures on $X$.)

THEOREM 1.2. *Let $X$ be a completely regular Hausdorff space. Then the following hold*:

1. *On $M_\sigma^+$ the topologies $e^b$ and $\sigma(M_\sigma, C^b)$ are identical.*
2. *If $B$ is a uniformly bounded, equicontinuous subset of $C^b$, then $B$ is relatively $\sigma(C^b, M_\sigma)$-compact.*

It is clear that statement 2 in Theorem 1.2 is an immediate consequence of statement 3 in Theorem 1.1 and the Mackey-Arens theorem. The proof of statement 1 is somewhat more complicated. However, by using Theorem 1.1, the analysis can be done essentially on the measures which have finite support. In this way a more elementary proof than that given by Granirer is obtained.

2. **The topology** $e$. A set $B \subset C$ is *pointwise bounded* if $\sup\{|f(x)| : f \in B\}$ is finite for all $x \in X$. Let $\mathscr{E}$ denote the family of all pointwise bounded, equicontinuous subsets of $C$. Let $M_c$ denote the subspace of $M_\sigma$ consisting of all those signed measures whose total variations have compact support in the realcompactification of $X$. (A Baire measure $m$ is said to have compact support in the realcompactification $vX$ of $X$ if there is a compact set $G$ in $vX$ such that $m(X - Z) = 0$ for every zero set $Z$ in $vX$ with $G \subset Z$.) Then $M_c$ and $C$ may be paired by the bilinear form $\langle m, f\rangle = \int_X f\, dm = \int_X f\, dm^+ - \int_X f\, dm^-$. It can be verified that the topology $e$ of uniform convergence on the sets in $\mathscr{E}$ is a locally convex topology on $M_c$. The following can then be demonstrated.

THEOREM 2.1. *Let $X$ be a completely regular Hausdorff space, and let $M_c$ be endowed with the topology $e$. Then the following hold*:

1. *$M_c$ is complete.*
2. *$L$ is dense in $M_c$.*
3. *The dual space of $M_c$ is $C$.*
4. *The restriction of the topology $e$ to $X$ is the original topology on $X$.*

In order to prove the theorem, one first shows that the topology $e$ on $L$ is the finest locally convex topology on $L$ whose restriction to $X$ is the original topology and that the dual space of $(L, e)$ is $C$. Again by combining Grothendieck's characterization of complete locally convex topologies with results from the theory of measures on topological spaces, one proves that the completion of $(L, e)$ is $(M_c, e)$.

An application of Theorem 2.1 yields the following analogue of Granirer's theorem (Theorem 1.2). The same method of proof is used here.

THEOREM 2.2. *Let $X$ be a completely regular Hausdorff space. Then the following hold*:

1. *On $M_c^+$ the topologies $e$ and $\sigma(M_c, C)$ are identical.*
2. *If $B$ is a pointwise bounded, equicontinuous subset of $C$, then $B$ is relatively $\sigma(C, M_c)$-compact.*

Another application of Theorem 2.1 yields a simple proof of the following theorem of Shirota. (See [**1**, p. 229].)

THEOREM 2.3. *Let $X$ be completely regular Hausdorff. Then there is a complete uniform structure compatible with the topology of $X$ if and only if $X$ is realcompact.*

The same methods can be used to explore the closure of $L$ in other natural topologies. For instance, the completion of $L$ for the topology of uniform convergence on the norm compact subsets of $C^b$ is the space $M$. Also if $M_\sigma(M_c)$ is given the Mackey topology for the pair $(M_\sigma, C^b)$ $((M_c, C))$, then it is complete and $L$ is a dense subspace. Furthermore, if $X$ is realcompact, then the completion of $L$ with the Mackey topology of the pair $(L, C)$ is $M_c$.

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