

GEOMETRY OF LEBESGUE-BOCHNER FUNCTION SPACES—SMOOTHNESS

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ABSTRACT. There exist real Banach spaces E such that the norm in E is of class C^∞ away from zero; however, for any p , $1 \leq p \leq \infty$, the norm in the Lebesgue-Bochner function space $L_p(E, \mu)$ is not even twice differentiable away from zero. It is this fact that led to a deeper study of the order of differentiability of the norm function in the spaces $L_p(E, \mu)$, and the main objective of this paper is to announce the complete determination of the order of smoothness of the norm in this class of Banach spaces.

1. Introduction. The class of Lebesgue-Bochner function spaces is discussed in Bochner and Taylor [1] and has been found to be of considerable importance in various branches of analysis. For their importance in Fourier analysis, the reader is referred to [1], and Stein and Weiss [11]. Various geometric properties of these spaces have been discussed in Day [5], Köthe [9], and McShane [10]; however, there has been no discussion of higher order smoothness of the norm in this class of Banach spaces. The only known result concerning smoothness is due to McShane [10], and his result concerns only the directional derivative (Gâteaux derivative) of the norm. In this paper, a complete characterization of k -times (continuous) differentiability of the norm in $L_p(E, \mu)$ is provided. It might be noted here that the order of smoothness of the Banach spaces $L_p(\mu)$ has been discussed in Sundaresan [12], and related results can be found in Bonic and Frampton [2].

2. Definitions and notation. The definitions and notation used throughout the paper are collected in this section for easy reference. In the following, E denotes a real Banach space. The unit ball of E is $U = \{x \in E \mid \|x\| \leq 1\}$, and its boundary $S = \{x \in E \mid \|x\| = 1\}$ is the unit sphere of E . In what follows, (T, Σ, μ) is an arbitrary measure space, where μ is an extended non-negative real-valued measure. To avoid trivialities it is assumed that the range of μ contains at least one nonzero real number, and μ is not supported by finitely many atoms.

2.1. DEFINITION. If $1 \leq p < \infty$, let

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$$\mathcal{L}_p(E, \mu) = \left\{ f \mid f: T \rightarrow E \text{ measurable, and } \int_T \|f(t)\|^p d\mu(t) < \infty \right\}.$$

Identifying functions in $\mathcal{L}_p(E, \mu)$ which agree μ -a.e., and equipping the resulting linear space with the norm $\|f\| = (\int_T \|f(t)\|^p d\mu(t))^{1/p}$, the *Lebesgue-Bochner function spaces* $L_p(E, \mu)$ are obtained.

For an account of these Banach spaces, see Bochner and Taylor [1], Dinculeanu [7], and Edwards [8].

2.2. DEFINITION. Let E and F be Banach spaces, and let $\mathcal{L}(E, F)$ denote the Banach space of *continuous linear mappings* on E into F with the usual operator norm. $\mathcal{B}^k(E, F)$ denotes the Banach space of continuous *k-multilinear mappings* $v: E \times E \times \dots \times E \rightarrow F$ with the norm

$$\|v\| = \sup_{\|x_i\| = \dots = \|x_k\| = 1} \|v(x_1, \dots, x_k)\|.$$

The spaces $\mathcal{B}^k(E, F)$ may be identified with the spaces defined inductively as follows:

$$\mathcal{B}^0(E, F) = F, \quad \mathcal{B}^k(E, F) = \mathcal{L}(E, \mathcal{B}^{k-1}(E, F)) = \mathcal{B}^{k-1}(E, \mathcal{L}(E, F)).$$

2.3. DEFINITION. Let E and F be Banach spaces, and let A be an open subset of E . A mapping $f: A \rightarrow F$ is said to be *differentiable at* $x \in A$ if there exists a mapping $f'(x) \in \mathcal{L}(E, F)$ such that

$$\lim_{h \rightarrow 0} \|f(x + h) - f(x) - f'(x) \cdot h\| / \|h\| = 0.$$

In this case f is continuous at $x \in A$ and $f'(x)$, which is unique, is called the *derivative of f at x* . The higher order derivatives $f^{(k)}: A \rightarrow \mathcal{B}^k(E, F)$ are defined in the usual manner (see Cartan [3] or Dieudonné [6]). The mapping $f: A \rightarrow F$ is said to be of *class C^k* or *k-times continuously differentiable* if it is *k-times differentiable* and the *kth-derivative* $f^{(k)}: A \rightarrow \mathcal{B}^k(E, F)$ is continuous.

The mapping $f: A \rightarrow F$ is said to be of *class C^∞* or *indefinitely continuously differentiable* if it is indefinitely differentiable.

From the results in Cudia [4], it follows that if the norm in a Banach space is differentiable away from zero, it is continuously differentiable away from zero.

2.4. REMARK. (i) Let E and F be Banach spaces and let A be an open subset of E . If the mapping $f: A \rightarrow F$ is *k-times differentiable* on A , then the *k-multilinear mapping* $f^{(k)}(x) \in \mathcal{B}^k(E, F)$ is symmetric for each $x \in A$.

(ii) Any continuous *k-linear mapping* is indefinitely differentiable, and all its derivatives of order $\geq k + 1$ are zero.

3. **Summary of results.** The order of smoothness of the Lebesgue-Bochner function spaces $L_p(E, \mu)$ is summarized in the following theorems.

3.1. THEOREM. *If $1 < p < \infty$, then the norm in $L_p(E, \mu)$ is differentiable (hence continuously differentiable) away from zero if and only if the norm in E is differentiable away from zero.*

3.2. THEOREM. *If $p > k \geq 2$, then the norm in $L_p(E, \mu)$ is k -times (continuously) differentiable away from zero if and only if the norm in E is k -times (continuously) differentiable away from zero and the k th derivative of the norm in E is uniformly bounded on the unit sphere in E .*

3.3. THEOREM. (i) *If $p > 2$ is an odd integer, then the norm in $L_p(E, \mu)$ is $(p - 1)$ -times (continuously) differentiable away from zero if and only if the norm in E is $(p - 1)$ -times (continuously) differentiable away from zero and the $(p - 1)$ st derivative of the norm in E is uniformly bounded on the unit sphere in E .*

(ii) *If $p > 2$ is not an integer and $I(p)$ is the integral part of p , then the norm in $L_p(E, \mu)$ is $I(p)$ -times (continuously) differentiable away from zero if and only if the norm in E is $I(p)$ -times (continuously) differentiable away from zero and the $I(p)$ th derivative of the norm in E is uniformly bounded on the unit sphere in E .*

3.4. REMARK. It follows from the results in [12], after noting that $L_p(\mu)$ is isometrically isomorphic to a subspace of $L_p(E, \mu)$, that if p is an odd integer, then the norm in $L_p(E, \mu)$ is never p -times differentiable away from zero.

The results are somewhat different when p is an even integer.

3.5. THEOREM. *If $p \geq 2$ is an even integer, then the norm in $L_p(E, \mu)$ is p -times (continuously) differentiable away from zero if and only if there exists a continuous homogeneous form T of degree p , $T: E \times E \times \cdots \times E \rightarrow \mathbf{R}$, such that $\|x\|^p = T(x, x, \dots, x)$ for all $x \in E$.*

There is a rather startling corollary to this theorem, namely

3.6. COROLLARY. *The norm in $L_2(E, \mu)$ is twice (continuously) differentiable away from zero if and only if E is a Hilbert space.*

Before proceeding to the counterexample mentioned in the abstract, a lemma is needed.

3.7. LEMMA. *If E is a Banach space whose norm is twice differentiable away from zero and the second derivative of the norm is uniformly bounded on the unit sphere in E , then E is uniformly smooth (in particular, E is reflexive).*

This is an immediate consequence of the generalized mean value theorem (see, e.g., Cartan [3, p. 70, Theorem 5.6.2]). This lemma together with Theorem 3.2 and Corollary 3.6 imply the following theorem:

3.8. THEOREM. *If $p \geq 2$ and the norm in $L_p(E, \mu)$ is twice differentiable away from zero, then E is reflexive. Indeed, E is uniformly smooth.*

4. **Example.** In this section an example is discussed, it is this example that led to a deeper study of the order of smoothness of the norm in $L_p(E, \mu)$, $1 \leq p \leq \infty$. Before proceeding to the example, a lemma is stated.

4.1. LEMMA [KUIPER]. *The Banach space c_0 is isomorphic to a Banach space whose norm is of class C^∞ away from zero.*

The proof of this lemma can be found on p. 896 of Bonic and Frampton [2].

4.2. EXAMPLE. Let E be the Banach space whose norm is of class C^∞ away from zero and which is isomorphic to c_0 . Let $1 \leq p \leq \infty$, then the norm in $L_p(E, \mu)$ is not even twice differentiable away from zero. The cases $p = 1$ or $p = \infty$ are trivial since in these cases the norm in $L_p(E, \mu)$ is not even once differentiable away from zero. In the case $1 < p < 2$, it follows from the results in [12] that the norm in $L_p(E, \mu)$ is not twice differentiable away from zero. Next let $2 \leq p < \infty$, and suppose the norm in $L_p(E, \mu)$ is twice differentiable away from zero. This implies by Theorem 3.8 that E is reflexive, hence c_0 is reflexive, a contradiction.

The proofs of the preceding theorems will appear elsewhere.

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