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ON THE DISCRETE REPRESENTATIONS OF THE GENERAL LINEAR GROUPS OVER A FINITE FIELD

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ABSTRACT. In this note we present a construction for a distinguished representation in the discrete series of $GL_n(F)$, F a finite field. This is used in describing explicitly Brauer's lifting of the identity representation of $GL_n(F)$.

Let V be a vector space of dimension $n \geqq 2$ over a finite field F with q elements. Let W_F be the ring of Witt vectors associated to F (see Serre [**4**,I, §6]) and K_F its field of fractions. Denote by $x \to \bar{x}$ the canonical ring homomorphism $W_F \to F$ and by $y \to \tilde{y}$ the canonical multiplicative homomorphism $F^* \to W_F$ such that $\tilde{y}^- = y$ for $y \in F^*$.

Our purpose is to construct explicitly a free W_F-module $D(V)$ associated canonically to V, which regarded as a representation of $GL(V)$ belongs to the discrete series, i.e. its character is a cusp form on $GL(V)$. We could call $D(V)$ the distinguished representation of the discrete series of $GL(V)$.

The construction is as follows (the details will appear elsewhere). Let X be the set of all sequences $(A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_{n-1})$ of affine subspaces of V (dim $A_i = i$) which are away from the origin, i.e. $0 \notin A_{n-1}$. Let $\mathscr{F}$ be the set of all functions $f: X \to W_F$. Consider the subset $\mathscr{F}' \subset \mathscr{F}$ consisting of all f's satisfying

(1) Given any fixed sequence $(A_0 \subset A_1 \subset \cdots \subset A_{i-1} \subset A_{i+1} \subset \cdots \subset A_{n-1})$ of affine subspaces of V away from the origin and a variable A_i between A_{i-1} and A_{i+1} away from the origin (there are q choices for A_i if $i = 0, n-1$ and $q+1$ choices if $0 < i < n-1$), we have

$$\sum_{A_i} f(A_0 \subset A_1 \subset \cdots \subset A_{i-1} \subset A_i \subset A_{i+1} \subset \cdots \subset A_{n-1}) = 0.$$

Define $\mathscr{F}'_1$ as the set of all $f \in \mathscr{F}'$ satisfying the homogeneity condition

(2) $f(\lambda A_0 \subset \lambda A_1 \subset \cdots \subset \lambda A_{n-1}) = \tilde{\lambda}^{-1} f(A_0 \subset A_1 \subset \cdots \subset A_{n-1})$, $\forall \lambda \in F^*$.

It is clear that $\mathscr{F}$, $\mathscr{F}'$, $\mathscr{F}'_1$ are finitely generated free W_F-modules. Define a W_F-linear map $t: \mathscr{F} \to \mathscr{F}$ by the formula

(3) $(tf)(A_0 \subset A_1 \subset \cdots \subset A_{n-1}) = (-1)^{n-1} \sum f(A'_0 \subset A'_1 \subset \cdots \subset A'_{n-1})$, where the sum is extended over all $(A'_0 \subset A'_1 \subset \cdots \subset A'_{n-1})$ in X such that $A'_0 \in A_{n-1} - A_{n-2}$, $A'_1 \| 0A_0$, $A'_2 \| 0A_1, \ldots, A'_{n-1} \| 0A_{n-2}$ (observe that once A'_0 is chosen, the A'_i's for $i > 0$ are automatically determined so that

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the number of terms in the sum equals $q^{n-1} - q^{n-2}$).[1]

One can show that t conserves the conditions (1) and (2), i.e. $t(\mathscr{F}'_{-1}) \subset \mathscr{F}'_{-1}$.

PROPOSITION 1. *The map* $t \otimes 1 : \mathscr{F}'_{-1} \otimes_{W_F} F \to \mathscr{F}'_{-1} \otimes_{W_F} F$ *is idempotent. All eigenvalues of* $t : \mathscr{F}'_{-1} \to \mathscr{F}'_{-1}$ *lie in* W_F. *There is precisely one eigenvalue* $\lambda(V) \in W_F$ *(repeated several times) such that* $\overline{\lambda(V)} = 1$.

Clearly $\lambda(V) \in W_F$ is an invariant of the vector space V.

PROPOSITION 2.
$$\lambda(V) = \sum_{y \in F'; \mathrm{trace}_{F'/F}\, y = 1} \tilde{y}^{-1}$$
where F' is an extension of degree n of the field F. (The sum has q^{n-1} terms which belong to $W_{F'}$, but after summing, the result lies in the subring $W_F \subset W_{F'}$.)

The fact that $\overline{\lambda(V)} = 1$ is contained in the following more general identity valid for integers k such that $k \equiv 1 (\mathrm{mod}\ q - 1)$:
$$\sum_{y \in F'; \mathrm{trace}_{F'/F}\, y = 1} y^{-k} = 1 \quad \text{if } k \equiv q^i (\mathrm{mod}\ q^n - 1) \text{ for some } i, \qquad = 0 \quad \text{otherwise.}$$

EXAMPLE. In case $n = 2$, $q = 2$ we have $\lambda(V) = -1$. In case $n = 2$, $q = 3$ we have $\lambda(V) = \sqrt{-2} - 1$ where $\sqrt{-2} \equiv -1$ (mod 3).

DEFINITION. $D(V) = \{f \in \mathscr{F}'_{-1} | tf = \lambda(V) f\}$.

If $\dim V = 1$ define $D(V) = \{f : V - 0 \to W_F | f(\lambda x) = \tilde{\lambda}^{-1} f(x),\ \lambda \in F^*,\ x \in V - 0\}$.

This is then a finitely generated free W_F-module which is a direct summand of $\mathscr{F}'_{-1}$. The general linear group $GL(V)$ operates naturally in $D(V)$ so that $D(V)$ becomes a representation space for $GL(V)$.

Next we describe some simplicial complexes associated to V. Given any partially ordered set S one can consider the simplicial complex whose k-simplices are precisely the totally ordered subsets of S having $k + 1$ elements.

EXAMPLES. (a) S = set of all affine subspaces of V away from the origin, ordered by inclusion. Let $A(V)$ be the corresponding simplicial complex.

(b) S = set of all proper linear subspaces of V which are transversal to a given proper linear subspace $V' \subset V$, ordered by inclusion. Let $T(V, V')$ be the corresponding simplicial complex.

(c) S = set of all affine subspaces of V strictly contained in a given hyperplane H in V (H away from the origin), ordered by inclusion. Let $C(H)$ be the corresponding simplicial complex. Note that $C(H)$ is canoni-

[1] The symbol $\|$ denotes "is parallel to."

cally isomorphic to $T(V, V')$ where V' is the unique hyperplane through the origin parallel to H.

(d) S = set of all proper linear subspaces of V ordered by inclusion. The corresponding symplicial complex is the well-known Tits complex $T(V)$ of V.

PROPOSITION 3. *Let $\tilde{H}$ denote reduced integral homology.* (a) $\tilde{H}_i(A(V)) = 0$ *for* $i \neq 0,\ n-1$, $\tilde{H}_{n-1}(A(V))$ *is free abelian of rank* $q^{n(n+1)/2}$ + *lower powers of* q.

(b) $\tilde{H}_i(T(V, V')) = 0$ *for* $i \neq 0,\ l-1$, $\tilde{H}_{l-1}(T(V, V'))$ *is free abelian of rank* $(q^{n-l}-1)(q^{n-l+1}-1)\cdots(q^{n-1}-1)$, $l = \dim V'$.

(c) $\tilde{H}_i(C(H)) = 0$ *for* $i \neq 0,\ n-2$, $\tilde{H}_{n-2}(C(H))$ *is free abelian of rank* $(q-1)(q^2-1)\cdots(q^{n-1}-1)$.

Note that (c) is a consequence of (b). It is easy to see that

$$H_{n-1}(A(V)) \otimes_Z W_F = \mathscr{F}.$$

Define a coefficient system (or sheaf) $\mathscr{G}$ over the Tits complex $T(V)$ as follows: To any simplex $\sigma = (V_{i_0} \subset V_{i_1} \subset \cdots \subset V_{i_k})$ of $T(V)$ (i.e., a flag of linear subspaces of V) we associated the vector space $\mathscr{G}_\sigma = V_{i_0}$. If $\sigma' = (V_{i_0} \subset \cdots \subset \hat{V}_{i_h} \subset \cdots \subset V_{i_k})$ is a face of σ, we have a natural map $\varphi_{\sigma\sigma'}: \mathscr{G}_\sigma \to \mathscr{G}_{\sigma'}$ defined as the identity $V_{i_0} \to V_{i_0}$ in case $h > 0$ or the natural inclusion $V_{i_0} \to V_{i_1}$ in case $h = 0$. It is clear that the system $(\mathscr{G}_0, \varphi_{\sigma\sigma'})$ form a coefficient system over $T(V)$ with respect to which one can consider simplicial homology.

PROPOSITION 4. *Assume* $n > 2$. *We have* $H_i(T(V); \mathscr{G}) = 0$ *for* $i \neq 0,\ n-2$, $H_0(T(V); \mathscr{G}) \cong V$, $H_{n-2}(T(V); \mathscr{G})$ = *F-vector space of dimension* $(q-1)(q^2-1)\cdots(q^{n-1}-1)$. *If* $n = 2$, *there is a natural surjective homomorphism* $H_0(T(V); \mathscr{G}) \to V$ *whose kernel has dimension* $q-1$ *and is canonically isomorphic to the space of homogeneous polynomials of degree* $q-2$ *on* V, *with values in* F.

Put

$$\begin{aligned} \tilde{H}_{n-2}(T(V); \mathscr{G}) &= \mathrm{H}_{n-2}(T(V); \mathscr{G}) && n > 2, \\ &= \ker(H_0(T(V); \mathscr{G}) \to V), && n = 2. \end{aligned}$$

PROPOSITION 5. *There is a canonical isomorphism* $D(V) \otimes_{W_F} F \cong \tilde{H}_{n-2}(T(V); \mathscr{G}) (n \geqq 2)$. *In particular* $\mathrm{rank}_{W_F} D(V) = (q-1)(q^2-1)\cdots(q^{n-1}-1)$.

PROPOSITION 6. *Choose an affine hyperplane* $H \subset V$ *away from the origin. There is a canonical isomorphism* (*depending on* H)

$$D(V) \cong \tilde{H}_{n-2}(C(H); W_F).$$

PROPOSITION 7. *Choose a proper linear subspace* $V' \subset V$, $\dim V' = l$. *There is a canonical isomorphism* (*depending on* V')

$$D(V) \otimes_{W_F} K_F \cong D(V/V') \otimes_{W_F} \tilde{H}_{l-1}(T(V, V'); K_F).$$

REMARKS. Proposition 5 shows that the F-reduction of $D(V)$ can be described homologically. One can prove that as soon as $n \geqq 3$ the modular representation $D(V) \otimes_{W_F} F$ of $GL(V)$ is not irreducible. It contains exactly 2 simple factors for $n = 3$. Proposition 6 identifies the restriction of $D(V)$ to the affine subgroup (i.e., the subgroup of all $\alpha \in GL(V)$ such that $\alpha(H) = H$) with a representation space described homologically. The homological description of the restriction of $D(V)$ to the affine subgroup has been also obtained independently by L. Solomon (to appear). It follows from results of S. Gel'fand [**1**] that this restriction is absolutely irreducible. This implies that $D(V)$ is an absolutely irreducible $GL(V)$-module.

Proposition 7 describes the restriction of $D(V)$ to any maximal parabolic subgroup of $GL(V)$. It is quite likely that the isomorphism of Proposition 7 holds also with K_F replaced by W_F (this is the case for $l = 1$ or $n - 1$).[2] Applying Proposition 6 repeatedly one can get a factorization into an iterated tensor product of the restriction of $D(V) \otimes_{W_F} K_F$ to any parabolic subgroup of $GL(V)$. For example the restriction to a Borel subgroup of $GL(V)$ is a tensor product of $n - 1$ representations of the Borel subgroup of dimensions $q - 1, q^2 - 1, \ldots, q^{n-1} - 1$.

PROPOSITION 8. *Define* $D^{(k)}(V) = \sum_{V' \subset V; 0 \in V'; \dim V' = k} D(V')$ $(1 \leqq k \leqq n)$. *Then* (a) *There is a canonical exact sequence*

$$0 \to D^{(n)}(V) \otimes_{W_F} F \to D^{(n-1)}(V) \otimes_{W_F} F \to \cdots \to D^{(1)}(V) \otimes_{W_F} F \to V \to 0.$$

(b) *Let* $V_1 \subset V$ *be a proper linear subspace of* V *and let* $\mathcal{U}(V_1) = \{\alpha \in GL(V): \alpha|V_1 = \text{identity}, \alpha|V/V_1 = \text{identity}\}$ *be the unipotent radical of the maximal parabolic subgroup corresponding to* V_1. *Decompose*

$$D^{(k)}(V) \otimes_{W_F} K_F = D^{(k)}_{\mathrm{I}}(V) \oplus D^{(k)}_{\mathrm{II}}(V)$$

where the first summand is the part on which $\mathcal{U}(V_1)$ *acts as identity and the second summand is the part on which* $\sum_{\alpha \in \mathcal{U}(V_1)} \alpha$ *acts as zero. Then there is a canonical exact sequence* (*depending on* V_1)

$$0 \to D^{(n)}_{\mathrm{II}}(V) \to D^{(n-1)}_{\mathrm{II}}(V) \to \cdots \to D^{(1)}_{\mathrm{II}}(V) \to 0.$$

Moreover $D^{(n)}_{\mathrm{I}}(V) = 0$.

(c) $D^{(k)}(V)$ *are absolutely irreducible.*

COROLLARY. *Let* $\beta: GL(V) \to W_F$ *be the character of the virtual representation* $D^{(1)}(V) - D^{(2)}(V) + \cdots + (-1)^{n-1} D^{(n)}(V)$. *Then for any* $\alpha \in$

[2] ADDED IN PROOF. This has been proved to be true in general.

$GL(V)$, $\beta(\alpha) = \Sigma\tilde{\lambda}$ *where the sum is over the n eigenvalues* λ *of* α.

REMARKS. 1. β is the classical Brauer lifting of the identity representation of $GL(V)$. Of course, in order to check that β was indeed a character, Brauer had to use his characterization of characters in terms of elementary subgroups, while here the virtual representation corresponding to β is constructed explicitly.

2. The complete description of the irreducible (complex) characters of the general linear group over a finite field is due to J. A. Green [**2**]. However explicit realizations of the discrete series representations were known only for SL_2 (see Tanaka [**5**]).

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