ON THE DISCRETE REPRESENTATIONS OF THE GENERAL LINEAR GROUPS OVER A FINITE FIELD

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Communicated by Michael F. Atiyah, October 9, 1972

ABSTRACT. In this note we present a construction for a distinguished representation in the discrete series of $GL_n(F)$, $F$ a finite field. This is used in describing explicitly Brauer's lifting of the identity representation of $GL_n(F)$.

Let $V$ be a vector space of dimension $n \geq 2$ over a finite field $F$ with $q$ elements. Let $W_F$ be the ring of Witt vectors associated to $F$ (see Serre [4, I, §6]) and $K_F$ its field of fractions. Denote by $x \mapsto \bar{x}$ the canonical ring homomorphism $W_F \to F$ and by $y \mapsto \hat{y}$ the canonical multiplicative homomorphism $F^* \to W_F$ such that $\hat{y}^z = y$ for $y \in F^*$.

Our purpose is to construct explicitly a free $W_F$-module $D(V)$ associated canonically to $K$ which regarded as a representation of $GL(V)$ belongs to the discrete series, i.e. its character is a cusp form on $GL(V)$. We could call $D(V)$ the distinguished representation of the discrete series of $GL(V)$.

The construction is as follows (the details will appear elsewhere). Let $X$ be the set of all sequences $(A_0 \subset A_1 \subset \cdots \subset A_{n-1})$ of affine subspaces of $V$ (dim $A_i = i$) which are away from the origin, i.e. $0 \notin A_{n-1}$. Let $\mathcal{F}$ be the set of all functions $f : X \to W_F$. Consider the subset $\mathcal{F}' \subset \mathcal{F}$ consisting of all $f$'s satisfying

1) Given any fixed sequence $(A_0 \subset A_1 \subset \cdots \subset A_{i-1} \subset A_{i+1} \subset \cdots \subset A_{n-1})$ of affine subspaces of $V$ away from the origin and a variable $A_i$ between $A_{i-1}$ and $A_{i+1}$ away from the origin (there are $q$ choices for $A_i$ if $i = 0, n - 1$ and $q + 1$ choices if $0 < i < n - 1$), we have

$$\sum_{A_i} f(A_0 \subset A_1 \subset \cdots \subset A_{i-1} \subset A_i \subset A_{i+1} \subset \cdots \subset A_{n-1}) = 0.$$ 

Define $\mathcal{F}'$ as the set of all $f \in \mathcal{F}'$ satisfying the homogeneity condition

2) $f(\lambda A_0 \subset \lambda A_1 \subset \cdots \subset \lambda A_{n-1}) = \lambda^{-1} f(A_0 \subset A_1 \subset \cdots \subset A_{n-1})$, $\forall \lambda \in F^*$.

It is clear that $\mathcal{F}$, $\mathcal{F}'$, $\mathcal{F}'$ are finitely generated free $W_F$-modules. Define a $W_F$-linear map $i : \mathcal{F}' \to \mathcal{F}$ by the formula

3) $i(f)(A_0 \subset A_1 \subset \cdots \subset A_{n-1}) = (-1)^{n-1} \sum f(A_0 \subset A_1 \subset \cdots \subset A'_{n-1})$, where the sum is extended over all $(A_0 \subset A_1 \subset \cdots \subset A'_{n-1})$ in $X$ such that $A_0 \in A_{n-1} - A_{n-2}$, $A_i \parallel A_0$, $A_0 \parallel A_{n-1}$, $A_i \parallel A_{n-1}$, $A_0 \parallel A_{n-2}$ (observe that once $A_0$ is chosen, the $A'_i$'s for $i > 0$ are automatically determined so that

the number of terms in the sum equals $q^{n-1} - q^{n-2})$.\(^1\)

One can show that $t$ conserves the conditions (1) and (2), i.e. $t(\mathcal{F}_1^-) \subset \mathcal{F}_1^-$.\(^1\)

**Proposition 1.** The map $t \otimes 1: \mathcal{F}_1^- \otimes_{W_F} F \to \mathcal{F}_1^- \otimes_{W_F} F$ is idempotent. All eigenvalues of $t: \mathcal{F}_1^- \to \mathcal{F}_1^-$ lie in $W_F$. There is precisely one eigenvalue $\lambda(V) \in W_F$ (repeated several times) such that $\lambda(V) = 1$.

Clearly $\lambda(V) \in W_F$ is an invariant of the vector space $V$.

**Proposition 2.**

$$\lambda(V) = \sum_{y \in F': \text{trace}_{F'/F} y = 1} y^{-1}$$

where $F'$ is an extension of degree $n$ of the field $F$. (The sum has $q^n - 1$ terms which belong to $W_{F'}$, but after summing, the result lies in the subring $W_F \subset W_{F'}$.)

The fact that $\lambda(V) = 1$ is contained in the following more general identity valid for integers $k$ such that $k \equiv 1(\text{mod } q - 1)$:

$$\sum_{y \in F': \text{trace}_{F'/F} y = 1} y^{-k} = 1 \text{ if } k \equiv q'(\text{mod } q^n - 1) \text{ for some } i, 0 = 0 \text{ otherwise.}$$

**Example.** In case $n = 2$, $q = 2$ we have $\lambda(V) = -1$. In case $n = 2$, $q = 3$ we have $\lambda(V) = \sqrt[n]{-2} - 1$ where $\sqrt[n]{-2} \equiv -1 \text{ (mod } 3).$

**Definition.** $D(V) = \{ f \in \mathcal{F}_1^- | tf = \lambda(V)f \}$. If $\dim V = 1$ define $D(V) = \{ f: V - 0 \to W_F | f(\lambda x) = \lambda^{-1} f(x), \lambda \in F^*, x \in V - 0 \}$. This is then a finitely generated free $W_F$-module which is a direct summand of $\mathcal{F}_1^-$. The general linear group $GL(V)$ operates naturally in $D(V)$ so that $D(V)$ becomes a representation space for $GL(V)$.

Next we describe some simplicial complexes associated to $V$. Given any partially ordered set $S$ one can consider the simplicial complex whose $k$-simplices are precisely the totally ordered subsets of $S$ having $k + 1$ elements.

**Examples.** (a) $S =$ set of all affine subspaces of $V$ away from the origin, ordered by inclusion. Let $A(V)$ be the corresponding simplicial complex.

(b) $S =$ set of all proper linear subspaces of $V$ which are transversal to a given proper linear subspace $V' \subset V$, ordered by inclusion. Let $T(V, V')$ be the corresponding simplicial complex.

(c) $S =$ set of all affine subspaces of $V$ strictly contained in a given hyperplane $H$ in $V$ ($H$ away from the origin), ordered by inclusion. Let $C(H)$ be the corresponding simplicial complex. Note that $C(H)$ is canonically parallel to $A(V)$.\(^1\)

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\(^1\) The symbol $\parallel$ denotes "is parallel to."
cally isomorphic to $T(V, V')$ where $V'$ is the unique hyperplane through the origin parallel to $H$.

(d) $S =$ set of all proper linear subspaces of $V$ ordered by inclusion. The corresponding simplicial complex is the well-known Tits complex $T(V)$ of $V$.

**Proposition 3.** Let $\tilde{H}$ denote reduced integral homology. (a) $\tilde{H}_i(A(V)) = 0$ for $i \neq 0$, $n - 1$, $\tilde{H}_{n-1}(A(V))$ is free abelian of rank $q^{(n+1)/2} + \text{lower powers of } q$.

(b) $\tilde{H}_i(T(V, V')) = 0$ for $i \neq 0$, $l - 1$, $\tilde{H}_{l-1}(T(V, V'))$ is free abelian of rank $(q^{n-1} - 1)(q^{n-1}+1 - 1) \cdots (q^{p-1} - 1)$, $l = \dim V'$.

(c) $\tilde{H}_i(C(H)) = 0$ for $i \neq 0$, $n - 2$, $\tilde{H}_{n-2}(C(H))$ is free abelian of rank $(q - 1)(q^2 - 1) \cdots (q^{n-1} - 1)$.

Note that (c) is a consequence of (b). It is easy to see that

$$H_{n-1}(A(V)) \otimes \mathbb{F} = \mathcal{F}.$$ Define a coefficient system (or sheaf) $\mathcal{G}$ over the Tits complex $T(V)$ as follows: To any simplex $\sigma = (V_{i_0} \subset V_{i_1} \subset \cdots \subset V_{i_k})$ of $T(V)$ (i.e., a flag of linear subspaces of $V$) we associated the vector space $\mathcal{G}_\sigma = V_{i_0}$. If $\sigma' = (V_{i_0} \subset \cdots \subset V_{i_h} \subset \cdots \subset V_{i_k})$ is a face of $\sigma$, we have a natural map $\varphi_{\sigma\sigma'}: \mathcal{G}_\sigma \to \mathcal{G}_{\sigma'}$ defined as the identity $V_{i_0} \to V_{i_0}$ in case $h > 0$ or the natural inclusion $V_{i_0} \to V_{i_1}$ in case $h = 0$. It is clear that the system $(\mathcal{G}_0, \varphi_{\sigma\sigma'})$ form a coefficient system over $T(V)$ with respect to which one can consider simplicial homology.

**Proposition 4.** Assume $n > 2$. We have $H_i(T(V); \mathcal{G}) = 0$ for $i \neq 0$, $n - 2$, $H_0(T(V); \mathcal{G}) \cong V$, $H_{n-2}(T(V); \mathcal{G})$ is $F$-vector space of dimension $(q - 1)(q^2 - 1) \cdots (q^{n-1} - 1)$. If $n = 2$, there is a natural surjective homomorphism $H_0(T(V); \mathcal{G}) \to V$ whose kernel has dimension $q - 1$ and is canonically isomorphic to the space of homogeneous polynomials of degree $q - 2$ on $V$, with values in $F$.

Put

$$\tilde{H}_{n-2}(T(V); \mathcal{G}) = H_{n-2}(T(V); \mathcal{G}) \quad n > 2,$$

$$= \ker(H_0(T(V); \mathcal{G}) \to V), \quad n = 2.$$ 

**Proposition 5.** There is a canonical isomorphism $D(V) \otimes_{\mathbb{F}} F \cong \tilde{H}_{n-2}(T(V); \mathcal{G})(n \geq 2)$. In particular $\text{rank}_{\mathbb{F}} D(V) = (q - 1)(q^2 - 1) \cdots (q^{n-1} - 1)$.

**Proposition 6.** Choose an affine hyperplane $H \subset V$ away from the origin. There is a canonical isomorphism (depending on $H$)

$$D(V) \cong \tilde{H}_{n-2}(C(H); \mathcal{G}).$$
Proposition 7. Choose a proper linear subspace \( V' \subset V \), \( \dim V' = l \).
There is a canonical isomorphism (depending on \( V' \))
\[
D(V) \otimes_{W_F} K_F \cong D(V/V') \otimes_{W_F} H_{l-1}(T(V, V'); K_F).
\]

Remarks. Proposition 5 shows that the \( F \)-reduction of \( D(V) \) can be described homologically. One can prove that as soon as \( n \geq 3 \) the modular representation \( D(V) \otimes_{W_F} F \) of \( GL(V) \) is not irreducible. It contains exactly 2 simple factors for \( n = 3 \). Proposition 6 identifies the restriction of \( D(V) \) to the affine subgroup (i.e., the subgroup of all \( \alpha \in GL(V) \) such that \( \alpha(H) = H \)) with a representation space described homologically. The homological description of the restriction of \( D(V) \) to the affine subgroup has been also obtained independently by L. Solomon (to appear). It follows from results of S. Gel'fand [1] that this restriction is absolutely irreducible. This implies that \( D(V) \) is an absolutely irreducible \( GL(V) \)-module.

Proposition 7 describes the restriction of \( D(V) \) to any maximal parabolic subgroup of \( GL(V) \). It is quite likely that the isomorphism of Proposition 7 holds also with \( K_F \) replaced by \( W_F \) (this is the case for \( l = 1 \) or \( n - 1 \)).

Applying Proposition 6 repeatedly one can get a factorization into an iterated tensor product of the restriction of \( D(V) \otimes_{W_F} K_F \) to any parabolic subgroup of \( GL(V) \). For example the restriction to a Borel subgroup of \( GL(V) \) is a tensor product of \( n - 1 \) representations of the Borel subgroup of dimensions \( q - 1, q^2 - 1, \ldots, q^{n-1} - 1 \).

Proposition 8. Define \( D^{(k)}(V) = \sum_{V : 0 \leq V : \dim V = k} D(V') \) (\( 1 \leq k \leq n \)). Then
(a) There is a canonical exact sequence
\[
0 \rightarrow D^{(n)}(V) \otimes_{W_F} F \rightarrow D^{(n-1)}(V) \otimes_{W_F} F \rightarrow \cdots \rightarrow D^{(1)}(V) \otimes_{W_F} F \rightarrow V \rightarrow 0.
\]
(b) Let \( V_1 \subset V \) be a proper linear subspace of \( V \) and let \( \mathcal{U}(V_1) = \{ \alpha \in GL(V) : \alpha|V_1 = \text{identity}, \alpha|V/V_1 = \text{identity} \} \) be the unipotent radical of the maximal parabolic subgroup corresponding to \( V_1 \). Decompose
\[
D^{(k)}(V) \otimes_{W_F} K_F = D^{(k)}_0(V) \oplus D^{(k)}_1(V)
\]
where the first summand is the part on which \( \mathcal{U}(V_1) \) acts as identity and the second summand is the part on which \( \sum_{\alpha \in \mathcal{U}(V_1)} \alpha \) acts as zero. Then there is a canonical exact sequence (depending on \( V_1 \))
\[
0 \rightarrow D^{(n)}_0(V) \rightarrow D^{(n-1)}_0(V) \rightarrow \cdots \rightarrow D^{(1)}_0(V) \rightarrow 0.
\]
Moreover \( D^{(n)}_0(V) = 0 \).
(c) \( D^{(k)}(V) \) are absolutely irreducible.

Corollary. Let \( \beta : GL(V) \rightarrow W_F \) be the character of the virtual representation \( D^{(1)}(V) - D^{(2)}(V) + \cdots + (-1)^{n-1} D^{(n)}(V) \). Then for any \( \alpha \in \mathcal{U}(V_1) \)
GL(V), \( \beta(\alpha) = \sum \lambda \) where the sum is over the \( n \) eigenvalues \( \lambda \) of \( \alpha \).

REMARKS. 1. \( \beta \) is the classical Brauer lifting of the identity representation of \( GL(V) \). Of course, in order to check that \( \beta \) was indeed a character, Brauer had to use his characterization of characters in terms of elementary subgroups, while here the virtual representation corresponding to \( \beta \) is constructed explicitly.

2. The complete description of the irreducible (complex) characters of the general linear group over a finite field is due to J. A. Green [2]. However explicit realizations of the discrete series representations were known only for \( SL_2 \) (see Tanaka [5]).

ACKNOWLEDGEMENTS. It is my pleasure to thank R. Carter for many stimulating conversations. I want to thank also M. Kervaire for his help in proving Proposition 2(i). My interest in understanding Brauer's lifting of modular characters has been aroused by Quillen's paper [3].

REFERENCES


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