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Φ-LIKE ANALYTIC FUNCTIONS. I

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The object of this paper is to introduce a very broad generalization, indeed a complete generalization of star-like and spiral-like functions. Our principal definition is the following.

DEFINITION 1. Let f be analytic in the unit disk $\Delta = \{z : |z| < 1\}$ of the complex plane with $f(0) = 0$, $f'(0) \neq 0$. Let Φ be analytic on $f(\Delta)$ with $\Phi(0) = 0$, $\operatorname{Re} \Phi'(0) > 0$. Then f *is* Φ-*like* (*in* Δ) if

$$\operatorname{Re}(zf'(z)/\Phi(f(z))) > 0 \qquad (z \in \Delta). \tag{1}$$

REMARKS. 1. The two classical cases of Definition (1) are given by $\Phi(w) = w$ (f is star-like) and, more generally, $\Phi(w) = \lambda w$, $\operatorname{Re} \lambda > 0$. ($f$ is spiral-like of type $\arg \lambda$.)

2. The conditions $\Phi(0) = 0$, $\operatorname{Re} \Phi'(0) > 0$ on Φ are necessary for the existence of an f as described satisfying (1). Conversely, if Φ, analytic in a neighborhood of 0, has these two properties, then there exist Φ-like functions f. For the present we mention only the trivial example $f(z) = az$, where $|a|$ is nonzero and sufficiently small.

3. In spite of the great generality of Definition 1, Φ-like functions are necessarily univalent in Δ (Theorem 1). Moreover the converse is true: Every function analytic and univalent in Δ and vanishing at 0 is Φ-like for some Φ (Corollary 1). Thus we shall obtain a characterization of univalence.

4. The definition immediately below will prove to be the geometric counterpart of Definition 1. (See Theorems 1 and 2.)

DEFINITION 2. Let Ω be a region containing 0, and let Φ be analytic on Ω with $\Phi(0) = 0$, $\operatorname{Re} \Phi'(0) > 0$. Then Ω *is* Φ-*like* if for any $\alpha \in \Omega$ the initial value problem

$$dw/dt = -\Phi(w), \qquad w(0) = \alpha \tag{2}$$

has a solution $w(t)$ defined for all $t \geqq 0$ such that $w(t) \in \Omega$ for all $t \geqq 0$ and $w(t) \to 0$ as $t \to +\infty$.

REMARKS. 5. If there is a solution of (2) on $[0, \infty)$, it is necessarily unique by a fundamental theorem on first order differential equations. For instance if $\alpha = 0$, then $w(t) = 0$ for all t.

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6. If $\Phi(w) = w$, the solution of (2) is $w(t) = \alpha e^{-t}$. Hence, for this Φ, Ω is Φ-like if and only if Ω is star-like with respect to 0. If $\Phi(w) = \lambda w$, $\operatorname{Re} \lambda > 0$, then $w(t) = \alpha e^{-\lambda t}$. Hence Ω is Φ-like if and only if Ω contains all spirals $\{\alpha e^{-\lambda t} : t \geqq 0\}$ joining points α of Ω to 0.

7. Perhaps the simplest example of a Φ other than those already mentioned is given by $\Phi(w) = w - w^2/\beta$ ($\beta \neq 0$). It is easy to see that if $\beta \in \Omega$ then Ω is not Φ-like. In fact a study shows that a necessary condition for Ω to be Φ-like is that Ω be disjoint from the ray $\{r\beta : r \geqq 1\}$. Moreover, if Ω fulfills this requirement, then Ω is Φ-like if and only if for each $\alpha \in \Omega$, the circular arc (or line segment) joining α to 0, concyclic with but not containing β, lies entirely in Ω.

8. For any Φ as described in Definition 2 each sufficiently small disk centered at 0 is Φ-like. Indeed, we can write $\Phi(w) = wp(w)$ where $\operatorname{Re} p(w) > 0$ for $|w|$ sufficiently small. Hence our assertion is one of the consequences of Lemma 1 below. Using terminology from the theory of differential equations, we can say that the origin is an asymptotically stable critical point of our differential equation $dw/dt = -\Phi(w)$.

LEMMA 1. *Let $p(z)$ be analytic for $|z| < r$ with* $\operatorname{Re} p(z) > 0$. *Then for any z with $|z| < r$, the initial value problem*

$$d\theta/dt = -\theta p(\theta), \qquad \theta(0) = z \tag{3}$$

has a solution defined for all $t \geqq 0$, and this solution approaches 0 *with nonincreasing modulus as $t \to +\infty$.*

PROOF. Let $|z| < r$. For $t \geqq 0$ and $n = 0, 1, 2, \ldots$ we define

$$\theta_0(t) = z, \qquad \theta_{n+1}(t) = z \exp\left[-\int_0^t p(\theta_n(x))\,dx\right]$$

noting that $|\theta_n(t)| \leqq |z| < r$ for all n and all t. Next we apply the inequality

$$|e^a - e^b| \leqq |a - b| \qquad (\operatorname{Re} a \leqq 0, \operatorname{Re} b \leqq 0)$$

to estimate $|\theta_{n+1}(t) - \theta_n(t)|$. It then follows in a familiar way (Picard iteration) that $\{\theta_n\}$ converges uniformly on any interval $[0, t], t \geqq 0$. Hence the limit function θ satisfies

$$\theta(t) = z \exp\left[-\int_0^t p(\theta(x))\,dx\right] \qquad (t \geqq 0)$$

and therefore (3). Clearly $|\theta(t)|$ is nonincreasing as t increases. Finally, since $|\theta(t)| \leqq |z|$, $\exists \delta > 0$ such that $\operatorname{Re} p(\theta(t)) \geqq \delta$ for $t \geqq 0$. Therefore

$$|\theta(t)| \leqq |z| e^{-\delta t} \to 0 \quad \text{as } t \to +\infty.$$

THEOREM 1. *Let f be Φ-like in Δ* (*Definition* 1). *Then f is univalent in Δ*

and $f(\Delta)$ *is* Φ*-like* (*Definition* 2).

PROOF. We define p by

$$p(z) = \Phi(f(z))/zf'(z) \qquad (z \in \Delta). \tag{4}$$

By (1), p is analytic in Δ with positive real part. Next we fix $z \in \Delta$ and define $\theta(t) = \theta_z(t)$ for $t \geqq 0$ by (3) and (4) (and Lemma 1). Finally we define $w(t) = w_z(t)$ by

$$w_z(t) = f(\theta_z(t)) \qquad (t \geqq 0). \tag{5}$$

Then an easy calculation based on (3), (4), and (5) shows that $w_z(t)$ is the solution for $t \geqq 0$ of

$$dw_z/dt = -\Phi(w_z), \qquad w_z(0) = f(z). \tag{6}$$

Moreover, by further use of Lemma 1 we obtain the result

$$\lim_{t\to+\infty} w_z(t) = \lim_{t\to+\infty} f(\theta_z(t)) = f(0) = 0.$$

It is now clear that $f(\Delta)$ is Φ-like.

To demonstrate the univalence of f we let $a, b \in \Delta$ and suppose $f(a) = f(b)$. In the notation of (5) and (6) we can write this as $w_a(0) = w_b(0)$. But then an application of the uniqueness theorem to (6) yields $w_a(t) = w_b(t)$ for all $t \geqq 0$. That is, $f(\theta_a(t)) = f(\theta_b(t))$ for $t \geqq 0$. Since $f'(0) \neq 0$ and since $\theta_a(t), \theta_b(t) \to 0$ as $t \to +\infty$, it follows that $\theta_a(t) = \theta_b(t)$ for t sufficiently large. By an application of the uniqueness theorem to (3) we conclude that $\theta_a(t) = \theta_b(t)$ for all $t \geqq 0$. Therefore $a = \theta_a(0) = \theta_b(0) = b$ as required.

COROLLARY 1. *Let f be analytic in Δ with $f(0) = 0$. Then f is univalent in Δ if and only if f is Φ-like for some Φ.*

PROOF. Suppose f is univalent in Δ. Let p be any function analytic in Δ with positive real part. Then the equation

$$\Phi(f(z)) = zf'(z)/p(z)$$

defines a function Φ, analytic in $f(\Delta)$ and satisfying (1). The converse is of course contained in Theorem 1.

REMARKS. 9. In the proof of Corollary 1 the following problem has been solved: Given f, univalent in Δ with $f(0) = 0$, find all Φ such that f is Φ-like in Δ. The converse problem is the following: Given Φ analytic in a neighborhood of 0 with $\Phi(0) = 0$ and $\operatorname{Re}\Phi'(0) > 0$, find all Φ-like functions f. We intend to discuss this matter in a second paper.

THEOREM 2. *Let f be analytic and univalent in Δ with $f(0) = 0$, and let $f(\Delta)$ be Φ-like* (*Definition* 2). *Then f is Φ-like in Δ* (*Definition* 1).

PROOF. We define $w_z(t)$ for $z \in \Delta$ and $t \geqq 0$ by (6). Next we define

$$\theta_z(t) = f^{-1}(w_z(t)) \qquad (z \in \Delta, t \geqq 0).$$

Then $f'(\theta_z(t))\theta_z'(t) = -\Phi(w_z(t))$ and $\theta_z'(0) = -\Phi(f(z))/f'(z)$. Since for $z = 0$, $\operatorname{Re} \Phi(f(z))/zf'(z) = \operatorname{Re} \Phi'(0) > 0$, we can complete the proof by showing that $\operatorname{Re} \theta_z'(0)/z \leqq 0$ for $z \in \Delta$, $z \neq 0$. For this we make some observations about $\theta_z(t)$. First it follows from (6) and a standard theorem on differential equations that $w_z(t)$ is analytic in z for each fixed $t \geqq 0$. Therefore the same is true of $\theta_z(t)$. Second, it is clear that $|\theta_z(t)| < 1$ for all $z \in \Delta$ and all $t \geqq 0$. Third, from (6) we obtain $w_0(0) = 0$. Therefore $w_0(t) = 0$ for all t by the uniqueness theorem. Hence $\theta_0(t) = 0$ for all $t \geqq 0$. We can now apply Schwarz's Lemma to conclude that $|\theta_z(t)| \leqq |z|$ for all $z \in \Delta$ and all $t \geqq 0$. Therefore

$$\operatorname{Re} \frac{\theta_z'(0)}{z} = \operatorname{Re} \lim_{t \to 0^+} \frac{\theta_z(t) - \theta_z(0)}{tz} = \lim_{t \to 0^+} \frac{1}{t} \operatorname{Re} \left[\frac{\theta_z(t)}{z} - 1 \right] \leqq 0$$

as required.

REMARKS. 10. In the above proof we have used ideas from Theorem 1 of [**1**]. (See also the original paper [**2**].) By exploiting this theorem fully we could have obtained the following stronger but more technical result: Let f be analytic in Δ with $f(0) = 0$ and $f'(0) \neq 0$. Let Φ be analytic on $f(\Delta)$ with $\Phi(0) = 0$ and $\operatorname{Re} \Phi'(0) > 0$. Suppose that for each r, $0 < r < 1$, there exists $\delta > 0$ such that (6) has a solution $w_z(t)$ defined for $0 \leqq t \leqq \delta$ and $|z| < r$. Suppose further that this solution satisfies the subordination relation $w_z(t) \prec f(z)$ in $|z| < r$ for each fixed t. Then f is Φ-like in Δ.

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