LINK MANIFOLDS AND PERIODICITY

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1. **Introduction.** This research was motivated by a number of considerations not the least of which was an interesting phenomenon of periodicity observed by Alan Durfee [1]. Letting M_k denote the Brieskorn variety $V(Z_0^k + Z_1^2 + Z_2^2 + \cdots + Z_n^2) \cap S^{2n+1}$ for $n \geq 3$, odd and $k = 1, 2, 3, \ldots$, he finds that $M_k \simeq M_{k+8}$ where $\simeq$ denotes diffeomorphism. It was pleasant to speculate on possible geometric explanations for this regularity.

We examine this in the context of transformation groups. M_k is an $O(n-1)$-manifold with three orbit types and orbit space D^4. The fixed point set corresponds to a torus link in $S^3 = \partial D^4$. Extending results of Hirzebruch and Erle [2] to the case of links we obtain a classification for these "link manifolds" in terms of invariants of links. These results are stated in §§2 and 3. §4 discusses periodicity. Proofs will appear elsewhere.

2. **Invariants of links.** Given a link $L \subset S^3$ together with assigned orientations for its components, one may form an oriented surface $F \subset S^3$ such that $\partial F = L$ and F induces the given orientation on L. The Seifert pairing is a bilinear map $\theta: H_1(F) \times H_1(F) \to Z$ given by $\theta(x, y) = l(i_* x, y)$ where i_* denotes "the push off of F in the direction of the positive normal" and $l(,)$ is the linking number in S^3. One then defines an even, symmetric bilinear form $f: H_1(F) \times H_1(F) \to Z$ by $f(x, y) = \theta(x, y) + \theta(y, x)$. It turns out that the signature of f is an invariant of link type. In fact, it is identical with the Murasugi signature of the link [3]. We denote it by $\sigma(L)$. Let $Af: H_1(F) \to \text{Hom}(H_1(F), Z)$ be the adjoint map $Af(x)(y) = f(x, y)$ and let $G(L) = \text{cokernel } (Af)$. Then $G(L)$ is also an invariant of L; it is the first homology group of the double branched cover of S^3 with branch set L.

Let $\tau G(L)$ be the torsion subgroup. Then there is a quadratic form $q(L): \tau G(L) \to Q/Z$. The equivalence class of this form is an invariant of L; it may be identified with the linking form of the double branched cover.

3. **$O(n)$-manifolds.**

DEFINITION. A *link-manifold* M^{2n+1} is a smooth closed manifold admitting a smooth action of the orthogonal group $O(n)$ such that all isotropy subgroups are conjugate to $O(n)$, $O(n-1)$ or $O(n-2)$, the orbit space is diffeomorphic to $D^4 = \{x \in R^4 | \|x\| \leq 1\}$ and the fixed point set in M corresponds to a link $L \subset S^3 = \partial D^4$.

Let B_{2n+2} denote the set of diffeomorphism classes of $O(n)$ link manifolds.

DEFINITION. $BP_{2n+2} \equiv$ the set of diffeomorphism classes of $(n - 1)$ connected $(2n + 1)$-manifolds which bound parallelizable manifolds.

THEOREM 1. (a) $B_{4k} \equiv BP_{4k}$ *for* $k > 1$.
(b) *Suppose* $M_1, M_2 \in B_{4k}, k > 1$, *and* $M_i \leftrightarrow L_i \subset S^3$.

$$G(L_1) \simeq G(L_2), \qquad q(L_1) \simeq q(L_2), \qquad \sigma(L_1) \geqq \sigma(L_2).$$

Then

$$M_1 \simeq M_2 \,\#\, \tfrac{1}{8}(\sigma(L_1) - \sigma(L_2)) \cdot \Sigma$$

where $\#$ denotes connected sum and Σ is the Milnor sphere.

REMARK. A link manifold automatically determines an orientation for the corresponding link so that the above invariants are defined.

It is *not* true that $B_{2(2k+1)} = BP_{2(2k+1)}$. However $B_{2(2k+1)} \subset BP_{2(2k+1)}$ and these manifolds may also be classified, this time in terms of Z_2 quadratic forms for the links. We omit discussion of these quadratic forms and only state

THEOREM 2. *Let n be odd, $n \geqq 3$, $T =$ tangent sphere bundle to S^n, $\Sigma_1 =$ Kervaire Sphere of dimension $2n - 1$. Then B_{2n} consists precisely of the manifolds*

$$(S^{n-1} \times S^n) \,\#\, (S^{n-1} \times S^n) \,\#\, \cdots \,\#\, (S^{n-1} \times S^n),$$

$$T \,\#\, T \,\#\, \cdots \,\#\, T,$$

$$(S^{n-1} \times S^n) \,\#\, \cdots \,\#\, (S^{n-1} \times S^n) \,\#\, \Sigma_1,$$

$$S^{2n-1}, \Sigma_1.$$

Manifolds corresponding to *knots* are just Σ_1 or S^{2n-1}. If a link has $r + 1$ components then T or $S^{n-1} \times S^n$ will appear r times in the connected sum. Mod 2 quadratic forms for the link detect the appearance of $S^{n-1} \times S^n$, T and Σ_1.

4. **Periodicity.** Given a link $L \subset S^3$ and a connected oriented surface $F \subset S^3$ with $\partial F = L$, the surface F may be represented as a disk with attached bands. Given two bands on the surface one may imagine passing one band through the other replacing an over-crossing by an under-crossing (see Figure 1). The same operation may be performed on a single band as in Figure 2. The result is a new surface and a new link. We say that L is *pass-equivalent* to L' ($L \sim L'$) if L' is obtained from L by a sequence of such band-passes.

Fact. If $M_1, M_2 \in B_{2n}$ for n odd $\geqq 3$ with $M_i \leftrightarrow L_i \subset S^3$ and $L_1 \sim L_2$, then $M_1 \simeq M_2$.
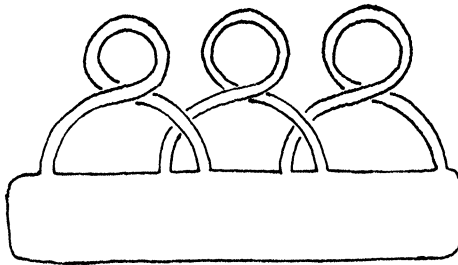
FIGURE 1
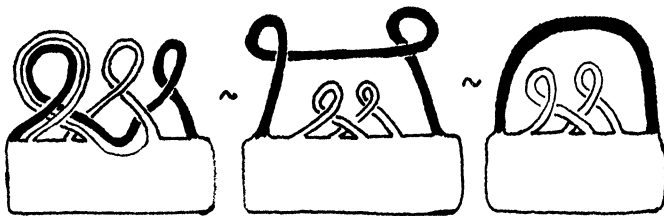

FIGURE 2

FIGURE 3

FIGURE 4

REMARK. This follows from [2]. M_1 and M_2 may be obtained by explicit surgery on a sphere corresponding to the band representations for the spanning surfaces of the links. Band-passing amounts to certain isotopies of the handles.

Our strategy for exhibiting the periodicity is to begin with a $(2, k)$-torus link having appropriate spanning surface and then simplify it by pass-equivalence. Figure 3 indicates a $(2, 4)$-link with spanning surface. In Figure 4 we show an embedded surface isotopic to that of Figure 3 and a sequence of pass-equivalences. These show that the $(2, 4)$-link is $\sim$ a $(2, 3)$-link connected sum with a trivial link of two circles. This then implies that $M_4 \simeq M_3 \# S^{n-1} \times S^n$. Similarly, one finds that $M_1 \simeq S^{2n-1}$, $M_2 \simeq T$, $M_3 \simeq \Sigma_1$ and

$$M_{2k+2} \simeq M_{2k+1} \# S^{n-1} \times S^n, \quad k \text{ odd},$$
$$\simeq M_{2k+1} \# T, \quad k \text{ even},$$
$$M_{2k+3} \simeq M_{2k+1}, \quad k \text{ odd},$$
$$\simeq M_{2k+1} \# \Sigma_1, \quad k \text{ even}.$$

Thus

$$M_1 \simeq S^{2n-1}, \qquad\qquad M_2 \simeq T,$$
$$M_3 \simeq \Sigma_1, \qquad\qquad M_4 \simeq \Sigma_1 \# S^{n-1} \times S^n,$$
$$M_5 \simeq \Sigma_1, \qquad\qquad M_6 \simeq \Sigma_1 \# T \simeq T,$$
$$M_7 \simeq \Sigma_1 \# \Sigma_1 \simeq S^{2n-1}, \qquad M_8 \simeq S^{n-1} \times S^n,$$

and since $M_9 \simeq S^{2n-1}$ the list begins repeating and we conclude that $M_{k+8} \simeq M_k$.

REMARK. The diffeomorphisms $\Sigma_1 \# T \simeq T$ and $M_3 \# M_3 \simeq S^{2n-1}$ may also be proved by band-passing. Since $M \# M \simeq S^{2n-1}$ implies that M is homeomorphic to a sphere, this constitutes an "elementary" proof that the Brieskorn variety M_3 is homeomorphic to a sphere.

REFERENCES

1. A. Durfee, *Diffeomorphism classification of isolated hypersurface singularities*, Thesis, Cornell University, Ithaca, N.Y., 1971.

2. D. Erle, *Die quadratische Form eines Knotens und ein Satz über Knotenmannigfaltig-keiten*, J. Reine Angew. Math. 236 (1969), 174–217. MR 40 #2106.

3. K. Murasugi, *On a certain numerical invariant of link types*, Trans. Amer. Math. Soc. 117 (1965), 387–422. MR 30 #1506.

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