

CLASS NUMBERS OF DEFINITE QUATERNARY FORMS WITH NONSQUARE DISCRIMINANT

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1. Introduction. Let V be a definite quadratic space of dimension four over the field of rational numbers \mathcal{Q} . If the discriminant $\Delta(V)$ is square, then the number of classes of maximal integral lattices can be computed by means of the formulas given by the author in [6] and [7]. The purpose of this note is to announce analogous class number formulas for the case where $\Delta(V)$ is not square, V_p is isotropic for each finite prime p , and the norm of the fundamental unit of $K = \mathcal{Q}((\Delta(V))^{1/2})$ is -1 .

Let \mathfrak{M} denote the genus of maximal integral lattices of V , \mathfrak{I} the ideal-complex containing \mathfrak{M} . Let Δ denote the discriminant of \mathfrak{M} . Then \mathfrak{I} can also be described as the set of all maximal lattices having reduced determinant Δ [1, p. 87]. The formulas we present here are for the number of (proper) similitude classes in \mathfrak{I} . We denote this class number by H . The number of classes in \mathfrak{M} will be denoted by H_0 . The ideal-complex \mathfrak{I} decomposes into g^+ similitude genera, where g^+ = the number of strict genera of K [5, p. 338]. The similitude genus containing \mathfrak{M} has H_0 similitude classes. It follows that $H_0 \leq H$. Equality holds if and only if K has prime discriminant, since $g^+ = 1$ in that case (cf. Corollary below).

2. Preliminaries. Let C_V^+ denote the second Clifford algebra of V . Then $C_V^+ = \mathfrak{A}_K = \mathfrak{A} \otimes_{\mathcal{Q}} K$, where \mathfrak{A} is a definite quaternion algebra over \mathcal{Q} . Let $\alpha \mapsto \alpha^*$ be the canonical involution of \mathfrak{A}_K and $N: \mathfrak{A}_K \rightarrow K$ the (reduced) norm mapping. The conjugation $x \mapsto \bar{x}$ of K can be extended to a \mathcal{Q} -automorphism $\alpha \mapsto \bar{\alpha}$ of \mathfrak{A}_K so that \mathfrak{A} is its ring of fixed elements. Let W be the set of all α in \mathfrak{A}_K such that $\alpha = \bar{\alpha}^*$. Then W is a four-dimensional \mathcal{Q} -subspace of \mathfrak{A}_K and the restriction of N to W takes values in \mathcal{Q} . In this way W may be regarded as a quadratic space over \mathcal{Q} . We assume, without loss of generality, that V is positive definite. Then V is isometric to W . In particular, the condition that V_p is isotropic for every finite prime p is equivalent to the condition that \mathfrak{A} splits at every finite prime which splits in K . From this it follows, by a straightforward computation of local discriminants, that

$$(1) \quad \Delta = \Delta_K(p_1 \cdots p_e)^2,$$

where Δ_K is the discriminant of K and p_1, \dots, p_e are all the nonsplit finite

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primes of \mathfrak{A} which remain prime in K . We put $\delta = p_1 \cdots p_e$.

3. Statement of results. Let D denote the square-free kernel of Δ_K . For any positive integer m let $\lambda(m)$ denote the number of primes dividing m ; let $h(-m)$ denote the class number of the imaginary quadratic extension $\mathcal{Q}((-m)^{1/2})$. Denote the Minkowski-Siegel weight of \mathfrak{F} by $M(\mathfrak{F})$.

THEOREM 1. *Suppose that V satisfies the conditions stated in the Introduction. If D is odd and $\Delta > 5$, then*

$$(2) \quad H = M(\mathfrak{F}) + c_1 h(-D) + c_3 h(-3D) + \sum_{n|\delta, d|D} 2^{-\lambda(n) - \sigma(nd)} h(-nd) h(-nD/d),$$

where $nd > 3$, $d < D^{1/2}$ and

$$\begin{aligned} c_1 &= \frac{1}{8} \quad \text{if } 2 \nmid \delta, \\ &= \frac{3}{16} \quad \text{if } 2|\delta, \\ c_3 &= \frac{1}{6} \quad \text{if } 3 \nmid \delta, \\ &= \frac{5}{6} \quad \text{if } 3|\delta, D \equiv 1 \pmod{8}, \\ &= \frac{1}{3} \quad \text{if } 3|\delta, D \equiv 5 \pmod{8}, \end{aligned}$$

and if $D \equiv 1 \pmod{8}$,

$$\begin{aligned} \sigma(m) &= -2 \quad \text{if } m \equiv 3 \pmod{8}, \\ &= 0 \quad \text{if } m \equiv 7 \pmod{8}, \\ &= 2 \quad \text{if } m \equiv 1 \pmod{4}, \end{aligned}$$

while if $D \equiv 5 \pmod{8}$,

$$\begin{aligned} \sigma(m) &= 0 \quad \text{if } m \equiv 3 \pmod{4}, \\ &= 2 \quad \text{if } m \equiv 2 \pmod{4}, \\ &= 2 \quad \text{if } m \equiv 1 \pmod{4}, 2 \nmid \delta, \\ &= 3 \quad \text{if } m \equiv 1 \pmod{4}, 2|\delta. \end{aligned}$$

Furthermore,

$$(3) \quad M(\mathfrak{F}) = \frac{\prod_{p|\delta} (p^2 + 1)}{3 \cdot 2^{e+2} \left(\binom{D}{\frac{D}{2}} - 4\right)} \left[\sum_{m=1}^{(D-1)/2} \binom{D}{m} m \right],$$

where $\binom{D}{m}$ is the Kronecker symbol.

REMARK. Since the fundamental unit of K has norm equal to -1 , every

prime divisor of D is congruent to 1 (mod 4). Hence D itself must be congruent to 1 (mod 4).

COROLLARY. *Suppose that $\Delta = p$, a prime greater than 5. Then*

$$(4) \quad H_0 = H = \frac{\sum_{m=1}^{(p-1)/2} \binom{p}{m} m}{12\left(\binom{p}{2} - 4\right)} + \frac{h(-p)}{8} + \frac{h(-3p)}{6}.$$

REMARKS. 1. If $\Delta = p$, a prime, then (1) implies $\Delta_K = p$. In this case it is well known that the norm of the fundamental unit of K is -1 .

2. Tamagawa has shown, under the assumption of the Corollary, that $H = h(\mathfrak{A}_K)/h(K)$, where $h(\mathfrak{A}_K)$ is the ideal class number of \mathfrak{A}_K and $h(K)$ is the ideal class number of K . Combining this with Peters' formula for $h(\mathfrak{A}_K)$ [5, p. 363], we obtain another proof of (4).

3. In the classical terminology, (4) is a formula for the number of classes of integral quaternary forms of discriminant p .

THEOREM 2. *Suppose that V satisfies the conditions stated in the Introduction. If D is even, then*

$$(5) \quad H = M(\mathfrak{J}) + \frac{5}{8}h(-D) + c_2h(-D/2) + c_3h(-3D) + \sum_{n|\delta, d|D} c_{nd}2^{-\lambda(n) - \sigma(nd)}h(-nd)h(-nD/d),$$

where $nd > 3$, $d < D^{1/2}$ and

$$\begin{aligned} c_2 &= 0 \quad \text{if } D = 2, \\ &= \frac{3}{4} \quad \text{if } D \neq 2, \\ c_3 &= \frac{1}{6} \quad \text{if } 3 \nmid \delta, \\ &= \frac{7}{12} \quad \text{if } 3|\delta, \end{aligned}$$

and for $m > 3$,

$$\begin{aligned} c_m &= 5 \quad \text{if } m \equiv 3 \pmod{8}, \\ &= 1 \quad \text{if } m \equiv 7 \pmod{8}, \\ &= 3 \quad \text{if } m \equiv 1 \pmod{4}, \\ \sigma(m) &= 1 \quad \text{if } m \equiv 3 \pmod{8}, \\ &= 0 \quad \text{if } m \equiv 7 \pmod{8}, \\ &= 2 \quad \text{if } m \equiv 1 \pmod{4}. \end{aligned}$$

Furthermore,

$$\begin{aligned}
 (6) \quad M(\mathfrak{S}) &= \frac{\prod_{p|\delta} (p^2 + 1)}{3 \cdot 2^{e+3}} \left[\sum_{m=1}^{D/2} \binom{4D}{m} (D + ((-1)^{(m-1)/2} - 1)m) \right] \text{ if } D \neq 2, \\
 &= \prod_{p|\delta} \frac{(p^2 + 1)}{3 \cdot 2^{e+3}} \text{ if } D = 2.
 \end{aligned}$$

4. Outline of the proof. If V is regarded as the subspace of all elements in \mathfrak{A}_K fixed by $\alpha \mapsto \bar{\alpha}^*$, then the proper similitudes of V are all the mappings of the form $\mu \mapsto r\alpha\mu\bar{\alpha}^*$, where r, α are invertible elements of $\mathcal{Q}, \mathfrak{A}_K$, respectively. Using this description of the group of proper similitudes, we deduce that $H = t_\delta$, the type number of all orders of level δ in \mathfrak{A}_K [2, p. 130]. Denote the multiplicative groups of K, \mathfrak{A}_K by $K^\times, \mathfrak{A}_K^\times$, respectively, and their ideal groups by $J_K, J_{\mathfrak{A}_K}$, respectively. Put $G = \mathfrak{A}_K^\times/K^\times, G_A = J_{\mathfrak{A}_K}/J_K$. Then G_A acts transitively on the collection of orders of level δ by conjugation. Fix an order Ω of \mathfrak{A}_K of level δ and denote its isotropy group under the action of G_A by $G_{\bar{\Omega}}$. Then we have

$$t_\delta = \text{card}(G_{\bar{\Omega}} \backslash G_A / G).$$

Proceeding as in [6], we regard t_δ as the trace of the convolution operator $f \mapsto F_{\bar{\Omega}} * f$ on $L_2(G_{\bar{\Omega}} \backslash G_A / G)$, where $F_{\bar{\Omega}}$ is the characteristic function of $G_{\bar{\Omega}}$. We fix a representative s from each conjugacy class of G and denote the centralizer of s in G by $G(s)$. Applying the Selberg trace formula, we obtain

$$(7) \quad H = t_\delta = \sum_s \int_{G_A/G(s)} \psi_s(g') dg',$$

where $\psi_s(g') = F_{\bar{\Omega}}(gsg^{-1})$ for g in G_A .

NOTATION. If α is an element of \mathfrak{A}_K and x is an algebraic number, then $\alpha \sim x$ will mean that α and x have the same minimal polynomial over K .

Making essential use of the conditions stated in the Introduction, we can show that a complete set of representatives for the conjugacy classes with nonzero contributions to (7) is given by $\{\alpha \text{ mod } K^\times\}$, where

- (i) $\alpha \sim 1$;
- (ii) $\alpha \sim \sqrt{-1}, \zeta$, where ζ is a primitive cube root of unity;
- (iii) $\alpha \sim 1 + \sqrt{-1}, \sqrt{-2}$ if $2|\Delta$;
- (iv) $\alpha \sim \sqrt{-3}, \zeta\sqrt{-3}$ if $3|\Delta$;
- (v) $\alpha \sim \sqrt{-m}$ if $m|\delta D$.

The contribution of (i) is $M(\mathfrak{S})$, which is evaluated by means of Leopoldt's formula for $L(2, \chi)$, where $\chi(m) = (\Delta_K/m)$ [4, p. 135]. The resulting expression is simplified as in [3, §6] to yield (3) and (6). The contribution of each of the remaining α in (ii)–(v) is evaluated and found to be a simple rational multiple of the relative class number of $K(\alpha)$ over K . The classical formula of Bachmann for the class number of a bicyclic biquadratic imaginary

extension of \mathcal{Q} is then invoked to yield the final formulas.

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