BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 3, May 1973

CLASS NUMBERS OF DEFINITE QUATERNARY FORMS WITH NONSQUARE DISCRIMINANT

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Communicated by Nathan Jacobson, November 2, 1972

1. **Introduction.** Let V be a definite quadratic space of dimension four over the field of rational numbers Q. If the discriminant $\Delta(V)$ is square, then the number of classes of maximal integral lattices can be computed by means of the formulas given by the author in [**6**] and [**7**]. The purpose of this note is to announce analogous class number formulas for the case where $\Delta(V)$ is not square, V_p is isotropic for each finite prime p, and the norm of the fundamental unit of $K = Q((\Delta(V))^{1/2})$ is -1.

Let $\mathfrak{M}$ denote the genus of maximal integral lattices of V, $\mathfrak{J}$ the idealcomplex containing $\mathfrak{M}$. Let Δ denote the discriminant of $\mathfrak{M}$. Then $\mathfrak{J}$ can also be described as the set of all maximal lattices having reduced determinant Δ [**1**, p. 87]. The formulas we present here are for the number of (proper) similitude classes in $\mathfrak{J}$. We denote this class number by H. The number of classes in $\mathfrak{M}$ will be denoted by H_0. The idealcomplex $\mathfrak{J}$ decomposes into g^+ similitude genera, where $g^+ =$ the number of strict genera of K [**5**, p. 338]. The similitude genus containing $\mathfrak{M}$ has H_0 similitude classes. It follows that $H_0 \leqq H$. Equality holds if and only if K has prime discriminant, since $g^+ = 1$ in that case (cf. Corollary below).

2. **Preliminaries.** Let C_V^+ denote the second Clifford algebra of V. Then $C_V^+ = \mathfrak{A}_K = \mathfrak{A} \otimes_Q K$, where $\mathfrak{A}$ is a definite quaternion algebra over Q. Let $\alpha \mapsto \alpha^*$ be the canonical involution of $\mathfrak{A}_K$ and $N : \mathfrak{A}_K \to K$ the (reduced) norm mapping. The conjugation $x \mapsto \bar{x}$ of K can be extended to a Q-automorphism $\alpha \mapsto \bar{\alpha}$ of $\mathfrak{A}_K$ so that $\mathfrak{A}$ is its ring of fixed elements. Let W be the set of all α in $\mathfrak{A}_K$ such that $\alpha = \bar{\alpha}^*$. Then W is a four-dimensional Q-subspace of $\mathfrak{A}_K$ and the restriction of N to W takes values in Q. In this way W may be regarded as a quadratic space over Q. We assume, without loss of generality, that V is positive definite. Then V is isometric to W. In particular, the condition that V_p is isotropic for every finite prime p is equivalent to the condition that $\mathfrak{A}$ splits at every finite prime which splits in K. From this it follows, by a straightforward computation of local discriminants, that

$$(1) \qquad \Delta = \Delta_K(p_1 \cdots p_e)^2,$$

where Δ_K is the discriminant of K and $p_1, \ldots, p_e$ are all the nonsplit finite

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AMS (MOS) subject classifications (1970). Primary 10C05, 12A80; Secondary 12A85.

[1] Partially supported by NSF grant GP-25320.

primes of $\mathfrak{A}$ which remain prime in K. We put $\delta = p_1 \cdots p_e$.

3. **Statement of results.** Let D denote the square-free kernel of Δ_K. For any positive integer m let $\lambda(m)$ denote the number of primes dividing m; let $h(-m)$ denote the class number of the imaginary quadratic extension $Q((-m)^{1/2})$. Denote the Minkowski-Siegel weight of $\mathfrak{J}$ by $M(\mathfrak{J})$.

THEOREM 1. *Suppose that V satisfies the conditions stated in the Introduction. If D is odd and $\Delta > 5$, then*

$$H = M(\mathfrak{J}) + c_1 h(-D) + c_3 h(-3D) + \sum_{n|\delta,\, d|D} 2^{-\lambda(n)-\sigma(nd)} h(-nd)h(-nD/d), \tag{2}$$

where $nd > 3$, $d < D^{1/2}$ and

$$c_1 = \tfrac{1}{8} \quad \text{if } 2\nmid\delta, \qquad = \tfrac{3}{16} \quad \text{if } 2|\delta,$$

$$c_3 = \tfrac{1}{6} \quad \text{if } 3\nmid\delta, \qquad = \tfrac{5}{6} \quad \text{if } 3|\delta,\ D \equiv 1 \pmod 8, \qquad = \tfrac{1}{3} \quad \text{if } 3|\delta,\ D \equiv 5 \pmod 8,$$

and if $D \equiv 1 \pmod 8$,

$$\sigma(m) = -2 \quad \text{if } m \equiv 3 \pmod 8, \qquad = 0 \quad \text{if } m \equiv 7 \pmod 8, \qquad = 2 \quad \text{if } m \equiv 1 \pmod 4,$$

while if $D \equiv 5 \pmod 8$,

$$\sigma(m) = 0 \quad \text{if } m \equiv 3 \pmod 4, \qquad = 2 \quad \text{if } m \equiv 2 \pmod 4, \qquad = 2 \quad \text{if } m \equiv 1 \pmod 4,\ 2\nmid\delta, \qquad = 3 \quad \text{if } m \equiv 1 \pmod 4,\ 2|\delta.$$

Furthermore,

$$M(\mathfrak{J}) = \frac{\prod_{p|\delta}(p^2+1)}{3 \cdot 2^{e+2}\left(\left(\frac{D}{2}\right) - 4\right)} \left[\sum_{m=1}^{(D-1)/2} \left(\frac{D}{m}\right) m\right], \tag{3}$$

where $\left(\frac{D}{m}\right)$ is the Kronecker symbol.

REMARK. Since the fundamental unit of K has norm equal to -1, every prime divisor of D is congruent to 1 (mod 4). Hence D itself must be congruent to 1 (mod 4).

COROLLARY. *Suppose that* $\Delta = p$, *a prime greater than* 5. *Then*

$$H_0 = H = \frac{\sum_{m=1}^{(p-1)/2} \left(\frac{p}{m}\right) m}{12\left(\left(\frac{p}{2}\right) - 4\right)} + \frac{h(-p)}{8} + \frac{h(-3p)}{6}. \tag{4}$$

REMARKS. 1. If $\Delta = p$, a prime, then (1) implies $\Delta_K = p$. In this case it is well known that the norm of the fundamental unit of K is -1.

2. Tamagawa has shown, under the assumption of the Corollary, that $H = h(\mathfrak{A}_K)/h(K)$, where $h(\mathfrak{A}_K)$ is the ideal class number of $\mathfrak{A}_K$ and $h(K)$ is the ideal class number of K. Combining this with Peters' formula for $h(\mathfrak{A}_K)$ [**5**, p. 363], we obtain another proof of (4).

3. In the classical terminology, (4) is a formula for the number of classes of integral quaternary forms of discriminant p.

THEOREM 2. *Suppose that* V *satisfies the conditions stated in the Introduction. If* D *is even, then*

$$\begin{aligned} H &= M(\mathfrak{J}) + \tfrac{5}{8}h(-D) + c_2 h(-D/2) + c_3 h(-3D) \\ &\quad + \sum_{n|\delta,\, d|D} c_{nd} 2^{-\lambda(n)-\sigma(nd)} h(-nd)h(-nD/d), \end{aligned} \tag{5}$$

where $nd > 3$, $d < D^{1/2}$ *and*

$$\begin{aligned} c_2 &= 0 \quad \text{if } D = 2, \\ &= \tfrac{3}{4} \quad \text{if } D \neq 2, \end{aligned}$$

$$\begin{aligned} c_3 &= \tfrac{1}{6} \quad \text{if } 3 \nmid \delta, \\ &= \tfrac{7}{12} \quad \text{if } 3|\delta, \end{aligned}$$

and for $m > 3$,

$$\begin{aligned} c_m &= 5 \quad \text{if } m = 3 \pmod 8, \\ &= 1 \quad \text{if } m = 7 \pmod 8, \\ &= 3 \quad \text{if } m = 1 \pmod 4, \end{aligned}$$

$$\begin{aligned} \sigma(m) &= 1 \quad \text{if } m = 3 \pmod 8, \\ &= 0 \quad \text{if } m = 7 \pmod 8, \\ &= 2 \quad \text{if } m = 1 \pmod 4. \end{aligned}$$

Furthermore,

$$M(\mathfrak{I}) = \frac{\prod_{p|\delta}(p^2+1)}{3\cdot 2^{e+3}}\left[\sum_{m=1}^{D/2}\left(\frac{4D}{m}\right)(D+((-1)^{(m-1)/2}-1)m)\right] \quad \textit{if } D \neq 2,$$
$$(6) \qquad = \prod_{p|\delta}\frac{(p^2+1)}{3\cdot 2^{e+3}} \textit{ if } D = 2.$$

4. **Outline of the proof.** If V is regarded as the subspace of all elements in $\mathfrak{A}_K$ fixed by $\alpha \mapsto \bar{\alpha}^*$, then the proper similitudes of V are all the mappings of the form $\mu \mapsto r\alpha\mu\bar{\alpha}^*$, where r, α are invertible elements of $\boldsymbol{Q}, \mathfrak{A}_K$, respectively. Using this description of the group of proper similitudes, we deduce that $H = t_\delta$, the type number of all orders of level δ in $\mathfrak{A}_K$ [**2**, p. 130]. Denote the multiplicative groups of $K, \mathfrak{A}_K$ by $K^\times, \mathfrak{A}_K^\times$, respectively, and their ideal groups by $J_K, J_{\mathfrak{A}_K}$, respectively. Put $G = \mathfrak{A}_K^\times/K^\times, G_A = J_{\mathfrak{A}_K}/J_K$. Then G_A acts transitively on the collection of orders of level δ by conjugation. Fix an order Ω of $\mathfrak{A}_K$ of level δ and denote its isotropy group under the action of G_A by $G_{\tilde{\Omega}}$. Then we have

$$t_\delta = \operatorname{card}(G_{\tilde{\Omega}}\backslash G_A/G).$$

Proceeding as in [**6**], we regard t_δ as the trace of the convolution operator $f \mapsto F_{\tilde{\Omega}} * f$ on $L_2(G_{\tilde{\Omega}}\backslash G_A/G)$, where $F_{\tilde{\Omega}}$ is the characteristic function of $G_{\tilde{\Omega}}$. We fix a representative s from each conjugacy class of G and denote the centralizer of s in G by $G(s)$. Applying the Selberg trace formula, we obtain

$$(7) \qquad H = t_\delta = \sum_s \int_{G_A/G(s)} \psi_s(g')\,dg',$$

where $\psi_s(g') = F_{\tilde{\Omega}}(gsg^{-1})$ for g in G_A.

NOTATION. If α is an element of $\mathfrak{A}_K$ and x is an algebraic number, then $\alpha \sim x$ will mean that α and x have the same minimal polynomial over K.

Making essential use of the conditions stated in the Introduction, we can show that a complete set of representatives for the conjugacy classes with nonzero contributions to (7) is given by $\{\alpha \bmod K^\times\}$, where

(i) $\alpha \sim 1$;

(ii) $\alpha \sim \sqrt{-1}, \zeta$, where ζ is a primitive cube root of unity;

(iii) $\alpha \sim 1 + \sqrt{-1}, \sqrt{-2}$ if $2|\Delta$;

(iv) $\alpha \sim \sqrt{-3}, \zeta\sqrt{-3}$ if $3|\Delta$;

(v) $\alpha \sim \sqrt{-m}$ if $m|\delta D$.

The contribution of (i) is $M(\mathfrak{I})$, which is evaluated by means of Leopoldt's formula for $L(2, \chi)$, where $\chi(m) = (\Delta_K/m)$ [**4**, p. 135]. The resulting expression is simplified as in [**3**, §6] to yield (3) and (6). The contribution of each of the remaining α in (ii)–(v) is evaluated and found to be a simple rational multiple of the relative class number of $K(\alpha)$ over K. The classical formula of Bachmann for the class number of a bicyclic biquadratic imaginary extension of Q is then invoked to yield the final formulas.

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