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HOMOTOPY EQUIVALENCES WHICH ARE CELLULAR AT THE PRIME 2

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0. We consider complexes having the property that the link of each point has the homology of S^{n-1} with coefficients in $Z[1/\text{odd}]$. Such complexes are called homology n-manifolds at the prime 2. Henceforth, all such spaces will be assumed to be 4-connected.

We state some salient facts.

LEMMA 1[**a**]. *Let K^n be a 4-connected Poincaré duality space, v its Spivak normal fibration and $T(v)$ the corresponding Thom spectrum. There is a spectrum $\mathscr{W}(v)$ and a map $p:\mathscr{W}(v) \to T(v)$ such that v is fiberhomotopically equivalent to a PL bundle if and only if p admits a section $s:T(v) \to \mathscr{W}(v)$.*

We remind the reader of the construction of $\mathscr{W}(v)$ in §1 below, where we construct another spectrum $\mathscr{W}_{(2)}(v)$, with an obvious natural map $f:\mathscr{W}(v) \to \mathscr{W}_{(2)}(v)$ such that p factors as $\mathscr{W}(v) \to^f \mathscr{W}_{(2)}(v) \to^{p_{(2)}} T(v)$.

LEMMA 2. *If the Poincaré duality space K^n is also a homology manifold at the prime 2, then $p_{(2)}:\mathscr{W}_{(2)}(v) \to T(v)$ admits a section.*

Lemma 2 is a consequence of straightforward geometric facts concerning homology manifolds with coefficients, namely, that "general position theorems" of the right sort hold for these objects.

Now let G be the direct sum of countably many copies of Z_2.

LEMMA 3. *For the map $f:\mathscr{W}(v) \to \mathscr{W}_2(v)$, $\pi_i(f) = 0$, if $i \geqq 5$, $\neq 4k$. If $i = 4k \geqq 8$, then $\pi_i(f) = G$.*

Lemma 3 is an abbreviation of Theorem A below. The main consequences are

THEOREM 1. *Let M^n be a 4-connected Poincaré duality space which is a homology n-manifold at the prime 2. Then M^n has the homotopy type of a PL manifold provided a sequence of obstructions in $H^{4k}(M^n, G)$ vanish for all k such that $4k < n$.*

In reality, these obstructions are to be thought of as the Thom isomorphism images of the obstructions to lifting the section $s_{(2)}: T(v) \to \mathscr{W}_{(2)}(v)$ to a section $s:T(v) \to \mathscr{W}(v)$. Thus Theorem 1 is almost obvious by virtue of Lemmas 1, 2, 3. We only remark that if $n = 4k$, we do not need to worry

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about the last obstruction in $H^n(M^n, G)$; we shall still obtain a section s (although not necessarily a lifting of $s_{(2)}$) since $\pi_{4k}(p) = 0$ for $k \geqq 2$ (see Theorem A, §1 below).

A relative version of Theorem 1 seems, in fact, more interesting.

THEOREM 2. *Let M^n, M_0^n be 3-connected PL manifolds and let $f: M_0^n \to M^n$ be a homotopy equivalence such that the mapping cylinder $\mathscr{M}_f$ may be triangulated as a homology $(n+1)$ manifold-with-boundary at the prime 2 (consistent with the combinatorial structure of M_0^n, M^n). Then f is homotopic to a PL equivalence, provided a sequence of obstructions in $H^{4k-1}(M^n, G)$ vanish for $4k-1 < n$.*

The thrust of the theorems is this: We know that if one has an "absolute" or "relative" triangulation problem, i.e., finding a PL manifold (or a PL equivalence) realizing the homotopy type of a Poincaré complex (or a homotopy equivalence between PL manifolds) there are obstructions every other dimension, half with Z_2 coefficients, half with Z coefficients. If, however, the problem is already "solved" in the world of homology manifolds at the prime 2, then we need only consider obstructions every fourth dimension with coefficients in the 2-group G.

We remark that if we replace the prime 2 in the above theorems by the world of a set of primes S including 2, then the same theorems hold with G replaced by a subgroup having one generator for each prime not in S.

1. **Poincaré duality spaces at the prime** 2. A space X will be called a Poincaré duality space at the prime 2 (of formal dimension n) if it is a finite complex X so that if R^{n+k} is a regular neighborhood of X in S^{n+k}, k large, then the map $\partial R^{n+k} \to {}^{\subseteq} R^{n+k}$ has a fiber which is the homotopy type of a $(k-1)$-sphere at the prime 2 (i.e., neglecting odd torsion). In particular, homology manifolds at the prime 2 are Poincaré duality spaces at the prime 2. Hereafter, we use "$\mathrm{PD}_{(2)}$-space" to mean Poincaré duality space at the prime 2.

Let ξ denote a sequence of $(k-1)$-spherical fibrations $\xi^k \to B_k$, together with maps $\phi_k: B_k \to B_{k+1}$ covered by spherical-fibration maps $\psi_k: \xi^k \oplus \varepsilon^0 \to \xi^{k+1}$, where ε^0 denotes the trivial S^0-fibration.

If X is a $\mathrm{PD}_{(2)}$-space of dimension n, then a ξ-structure on X is a diagram

$$\begin{array}{ccc} \partial R^{n+k} & \to & E(\xi^k) \\ \cap\!\!\mid & & \cap\!\!\mid \\ R^{n+k} & \to & \mathscr{M}_{\xi} k \approx B_k \end{array}$$

where R^{n+k} is a regular neighborhood and the map is a map of spherical fibrations at the prime 2. Of course if such a diagram exists, for a certain k, then one exists with k replaced by $k+1, k+2, \ldots$ which is completely determined by the original one (i.e., we may suspend such a diagram). All these are to be thought of as representing the same ξ-structure. We may define pairs and bordisms similarly, and we see that we obtain a bordism group of such objects by the usual construction. Setting $T = T(\xi) =$ the spectrum $\{T(\xi^k)\}$, we denote this bordism group by $\Omega_*^{T,2}$. The Pontrjagin-Thom construction gives a map $\Omega_*^{T,2} \to \pi_* T$.

Now let ξ^k be a spherical fibration and $f: M^n \to T(\xi^k)$ where M^n is a manifold. Recall that $T(\xi^k) = \mathscr{M}_{\xi^k} \cup_{E(\xi^k)} cE(\xi^k)$ where $\mathscr{M}$ denotes mapping cylinder, E total space and c unreduced cone. We call f t-regular at 2 iff

(1) $U = f^{-1}(\mathscr{M}_{\xi^k})$, $V = f^{-1}(cE(\xi^k))$ are codimension 0 submanifolds of M^n with common boundary

$$\partial U = \partial V = W = f^{-1}(E(\xi^k))$$

with the diagram

$$\begin{array}{ccc} W & \to & E(\xi^k) \\ \cap\!\!\!| & & \cap\!\!\!| \\ U & \to & \mathscr{M}_{\xi^k} \end{array}$$

as a map of $(k-1)$-spherical fibrations at the prime 2 (or if $U = \varnothing$).

A similar definition may readily be made when M has a boundary.

Let Δ^j denote the standard j-simplex; let $\mathscr{W}_{(2)}(\xi^k)$ denote the subcomplex of the singular complex of $T(\xi^k)$ consisting of those singular simplices $\sigma: \Delta^j — T(\xi^k)$ which are t-regular in 2, as are all of their lower dimensional faces. Now if $T = T(\xi) = \{T(\xi^k)\}$ then $\{\mathscr{W}_{(2)}(\xi^k)\}$ form a spectrum $\mathscr{W}_{(2)}(T)$.

LEMMA. $\pi_* \mathscr{W}_{(2)}(T) = \lim \pi_{k+*} \mathscr{W}_{(2)}(\xi^k)$ *is naturally equal to* $\Omega_*^{T,2}$; *moreover, the natural map* $\mathscr{W}_{(2)}(T) \to T$ *induces the Pontrjagin-Thom map* $\Omega_*^{T,2} \to \pi_* T$.

Of course, if we work in the world of all primes, we may make all of the above definitions and constructions, simply by ignoring the phrase "at the prime 2." (See [**c**], [**d**].) In other words, we could talk about true PD-spaces with ξ-structures on their normal fibrations, and form bordism groups Ω_*^T and a spectrum $\mathscr{W}(T)$ with $\pi_* \mathscr{W}(T) = \Omega_*^T$. In that case $\mathscr{W}(T) \subseteqq \mathscr{W}_{(2)}(T)$. [**c**] and [**d**] study the map $p: \mathscr{W}(T) \to T$.

We then have the following proposition. (Assume from now on that all base spaces are 1-connected.)

THEOREM A. *Consider the diagram*

$$\begin{tikzcd} F \arrow[r, dash] \arrow[d] & F_{(2)} \arrow[d] \\ \mathscr{W}(T) \arrow[r] \arrow[d, "p"] & \mathscr{W}_{(2)}(T) \arrow[d, "p_{(2)}"] \\ T & T \end{tikzcd}$$

where F and $F_{(2)}$ are the fibers of p and $p_{(2)}$ respectively. Then

$$\begin{aligned}\pi_i(F) &= L_i(Z),\\ \pi_i(F_{(2)}) &= L_i(Z[1/\mathrm{odd}]) \quad \text{for } i \geqq 4.\end{aligned} \tag{1}$$

Here L_i denotes the Wall L-group of the rings Z and $Z[1/\mathrm{odd}]$, respectively.

The map $F \to F_2$ induces the same map on the L-groups as the ring inclusion $Z \to Z[1/\mathrm{odd}]$.

(2) $L_i(Z) \to L_i(Z[1/\mathrm{odd}])$ is an isomorphism for $i \neq 4k$; for $i = 4k$, it is a monomorphism with cokernel G.

Part (1) is essentially stated, along with a sketch of the proof, by L. Jones in [**e**]. The author also has an independent proof (for a slightly weaker result). (See Appendix, §2.)

Part (2) is based in a direct computation of $L_*(Z[1/\mathrm{odd}])[f]$.

2. **Appendix.** We propose here to give some idea of how Theorem A is proved or at least of why it should be plausible. We restrict our attention to the case when ξ is a sequence of PL bundles.

PROPOSITION. *Let X^n be a 1-connected $PD_{(2)}$-space with ξ-structure, $n \geqq 5$. Let f be the Pontrjagin-Thom map determined by the ξ-structure on X, $f: S^{n+k} \to T(\xi^k)$; and let $g: D^{n+k+1} \to T(\xi^k)$ be an extension of f. (So, in particular the image of $[X] \in \Omega_n^{T(\xi),2}$ is 0 in $\pi_n T(\xi)$.) Then g may be deformed so as to be t-regular at 2 if and only if a certain invariant in $L_n(Z[1/\mathrm{odd}])$ vanishes.*

PROOF. We may assume that on a collar $S^{n+k} \times I$ of S^{n+k} in D^{n+k+1}, g is just $f \circ$ (projection). Since ξ^k has a PL structure, we may deform f so as to be PL transverse regular, i.e., $f^{-1}(B_k)$ is an n-manifold M sitting inside R^{n+k} = regular neighborhood X. It turns out that there is a bundle map $\nu^k(M) \to \eta^*\xi^k$ where $\eta: X \to B_k$ is part of the ξ-structure of X. Moreover, M bounds W (by putting all of g into PL transverse position). There is a bundle map $\nu^k(W) \to \xi^k$.

Now assume W lives in the smaller copy of D^{n+k+1} got by removing the open collar on S^{n+k}, and that M lives on the "inside" copy of R^{n+k}. If $M \subseteqq X$ is a homotopy equivalence at 2, then it is easy to see that the pair $(R \times I \cup W, R)$ is a $\mathrm{PD}_{(2)}$-pair with ξ-structure (here R is essentially X). So the problem is to make $M \subseteqq X$ a homotopy equivalence at 2 by doing surgery on the bundle map $\nu^k(M) \to \eta^*\xi^k$ (which covers a degree-one-map of $\mathrm{PD}_{(2)}$-spaces). We run into a well-defined obstruction in $L_n(Z[1/\mathrm{odd}])$, which vanishes when the surgery can be completed [**i**]. Q.E.D.

This lemma is the key step in the author's proof of Theorem A. In fact, the lemma by itself provides most of what is needed for a proof of Theorem 2 (which does not really need Theorem A in full generality).

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