

## HOMOTOPY EQUIVALENCES WHICH ARE CELLULAR AT THE PRIME 2

BY N. LEVITT

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0. We consider complexes having the property that the link of each point has the homology of  $S^{n-1}$  with coefficients in  $Z[1/\text{odd}]$ . Such complexes are called homology  $n$ -manifolds at the prime 2. Henceforth, all such spaces will be assumed to be 4-connected.

We state some salient facts.

LEMMA 1[a]. *Let  $K^n$  be a 4-connected Poincaré duality space,  $v$  its Spivak normal fibration and  $T(v)$  the corresponding Thom spectrum. There is a spectrum  $\mathcal{W}(v)$  and a map  $p: \mathcal{W}(v) \rightarrow T(v)$  such that  $v$  is fiberhomotopically equivalent to a PL bundle if and only if  $p$  admits a section  $s: T(v) \rightarrow \mathcal{W}(v)$ .*

We remind the reader of the construction of  $\mathcal{W}(v)$  in §1 below, where we construct another spectrum  $\mathcal{W}_{(2)}(v)$ , with an obvious natural map  $f: \mathcal{W}(v) \rightarrow \mathcal{W}_{(2)}(v)$  such that  $p$  factors as  $\mathcal{W}(v) \xrightarrow{f} \mathcal{W}_{(2)}(v) \xrightarrow{p^{(2)}} T(v)$ .

LEMMA 2. *If the Poincaré duality space  $K^n$  is also a homology manifold at the prime 2, then  $p_{(2)}: \mathcal{W}_{(2)}(v) \rightarrow T(v)$  admits a section.*

Lemma 2 is a consequence of straightforward geometric facts concerning homology manifolds with coefficients, namely, that “general position theorems” of the right sort hold for these objects.

Now let  $G$  be the direct sum of countably many copies of  $Z_2$ .

LEMMA 3. *For the map  $f: \mathcal{W}(v) \rightarrow \mathcal{W}_{(2)}(v)$ ,  $\pi_i(f) = 0$ , if  $i \geq 5$ ,  $\neq 4k$ . If  $i = 4k \geq 8$ , then  $\pi_i(f) = G$ .*

Lemma 3 is an abbreviation of Theorem A below. The main consequences are

THEOREM 1. *Let  $M^n$  be a 4-connected Poincaré duality space which is a homology  $n$ -manifold at the prime 2. Then  $M^n$  has the homotopy type of a PL manifold provided a sequence of obstructions in  $H^{4k}(M^n, G)$  vanish for all  $k$  such that  $4k < n$ .*

In reality, these obstructions are to be thought of as the Thom isomorphism images of the obstructions to lifting the section  $s_{(2)}: T(v) \rightarrow \mathcal{W}_{(2)}(v)$  to a section  $s: T(v) \rightarrow \mathcal{W}(v)$ . Thus Theorem 1 is almost obvious by virtue of Lemmas 1, 2, 3. We only remark that if  $n = 4k$ , we do not need to worry

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about the last obstruction in  $H^n(M^n, G)$ ; we shall still obtain a section  $s$  (although not necessarily a lifting of  $s_{(2)}$ ) since  $\pi_{4k}(p) = 0$  for  $k \geq 2$  (see Theorem A, §1 below).

A relative version of Theorem 1 seems, in fact, more interesting.

**THEOREM 2.** *Let  $M^n, M_0^n$  be 3-connected PL manifolds and let  $f: M_0^n \rightarrow M^n$  be a homotopy equivalence such that the mapping cylinder  $\mathcal{M}_f$  may be triangulated as a homology  $(n + 1)$  manifold-with-boundary at the prime 2 (consistent with the combinatorial structure of  $M_0^n, M^n$ ). Then  $f$  is homotopic to a PL equivalence, provided a sequence of obstructions in  $H^{4k-1}(M^n, G)$  vanish for  $4k - 1 < n$ .*

The thrust of the theorems is this: We know that if one has an “absolute” or “relative” triangulation problem, i.e., finding a PL manifold (or a PL equivalence) realizing the homotopy type of a Poincaré complex (or a homotopy equivalence between PL manifolds) there are obstructions every other dimension, half with  $Z_2$  coefficients, half with  $Z$  coefficients. If, however, the problem is already “solved” in the world of homology manifolds at the prime 2, then we need only consider obstructions every fourth dimension with coefficients in the 2-group  $G$ .

We remark that if we replace the prime 2 in the above theorems by the world of a set of primes  $S$  including 2, then the same theorems hold with  $G$  replaced by a subgroup having one generator for each prime not in  $S$ .

**1. Poincaré duality spaces at the prime 2.** A space  $X$  will be called a Poincaré duality space at the prime 2 (of formal dimension  $n$ ) if it is a finite complex  $X$  so that if  $R^{n+k}$  is a regular neighborhood of  $X$  in  $S^{n+k}$ ,  $k$  large, then the map  $\partial R^{n+k} \rightarrow \varepsilon R^{n+k}$  has a fiber which is the homotopy type of a  $(k - 1)$ -sphere at the prime 2 (i.e., neglecting odd torsion). In particular, homology manifolds at the prime 2 are Poincaré duality spaces at the prime 2. Hereafter, we use “ $PD_{(2)}$ -space” to mean Poincaré duality space at the prime 2.

Let  $\xi$  denote a sequence of  $(k - 1)$ -spherical fibrations  $\xi^k \rightarrow B_k$ , together with maps  $\phi_k: B_k \rightarrow B_{k+1}$  covered by spherical-fibration maps  $\psi_k: \xi^k \oplus \varepsilon^0 \rightarrow \xi^{k+1}$ , where  $\varepsilon^0$  denotes the trivial  $S^0$ -fibration.

If  $X$  is a  $PD_{(2)}$ -space of dimension  $n$ , then a  $\xi$ -structure on  $X$  is a diagram

$$\begin{array}{ccc} \partial R^{n+k} & \rightarrow & E(\xi^k) \\ \text{I} \cap & & \text{I} \cap \\ R^{n+k} & \rightarrow & \mathcal{M}_\xi k \approx B_k \end{array}$$

where  $R^{n+k}$  is a regular neighborhood and the map is a map of spherical fibrations at the prime 2. Of course if such a diagram exists, for a certain  $k$ , then one exists with  $k$  replaced by  $k + 1, k + 2, \dots$  which is completely determined by the original one (i.e., we may suspend such a diagram). All

these are to be thought of as representing the same  $\xi$ -structure. We may define pairs and bordisms similarly, and we see that we obtain a bordism group of such objects by the usual construction. Setting  $T = T(\xi) =$  the spectrum  $\{T(\xi^k)\}$ , we denote this bordism group by  $\Omega_*^{T,2}$ . The Pontrjagin-Thom construction gives a map  $\Omega_*^{T,2} \rightarrow \pi_* T$ .

Now let  $\xi^k$  be a spherical fibration and  $f: M^n \rightarrow T(\xi^k)$  where  $M^n$  is a manifold. Recall that  $T(\xi^k) = \mathcal{M}_{\xi^k \cup E(\xi^k)} cE(\xi^k)$  where  $\mathcal{M}$  denotes mapping cylinder,  $E$  total space and  $c$  unreduced cone. We call  $f$   $t$ -regular at 2 iff

(1)  $U = f^{-1}(\mathcal{M}_{\xi^k}), V = f^{-1}(cE(\xi^k))$  are codimension 0 submanifolds of  $M^n$  with common boundary

$$\partial U = \partial V = W = f^{-1}(E(\xi^k))$$

with the diagram

$$\begin{array}{ccc} W & \rightarrow & E(\xi^k) \\ \cap & & \cap \\ U & \rightarrow & \mathcal{M}_{\xi^k} \end{array}$$

as a map of  $(k - 1)$ -spherical fibrations at the prime 2 (or if  $U = \emptyset$ ).

A similar definition may readily be made when  $M$  has a boundary.

Let  $\Delta^j$  denote the standard  $j$ -simplex; let  $\mathcal{W}_{(2)}(\xi^k)$  denote the subcomplex of the singular complex of  $T(\xi^k)$  consisting of those singular simplices  $\sigma: \Delta^j \rightarrow T(\xi^k)$  which are  $t$ -regular in 2, as are all of their lower dimensional faces. Now if  $T = T(\xi) = \{T(\xi^k)\}$  then  $\{\mathcal{W}_{(2)}(\xi^k)\}$  form a spectrum  $\mathcal{W}_{(2)}(T)$ .

LEMMA.  $\pi_* \mathcal{W}_{(2)}(T) = \lim \pi_{k+*} \mathcal{W}_{(2)}(\xi^k)$  is naturally equal to  $\Omega_*^{T,2}$ ; moreover, the natural map  $\mathcal{W}_{(2)}(T) \rightarrow T$  induces the Pontrjagin-Thom map  $\Omega_*^{T,2} \rightarrow \pi_* T$ .

Of course, if we work in the world of all primes, we may make all of the above definitions and constructions, simply by ignoring the phrase ‘‘at the prime 2.’’ (See [c], [d].) In other words, we could talk about true PD-spaces with  $\xi$ -structures on their normal fibrations, and form bordism groups  $\Omega_*^T$  and a spectrum  $\mathcal{W}(T)$  with  $\pi_* \mathcal{W}(T) = \Omega_*^T$ . In that case  $\mathcal{W}(T) \cong \mathcal{W}_{(2)}(T)$ . [c] and [d] study the map  $p: \mathcal{W}(T) \rightarrow T$ .

We then have the following proposition. (Assume from now on that all base spaces are 1-connected.)

THEOREM A. Consider the diagram

$$\begin{array}{ccc} F & \longrightarrow & F_{(2)} \\ \downarrow & & \downarrow \\ \mathcal{W}(T) & \longrightarrow & \mathcal{W}_{(2)}(T) \\ \downarrow p & & \downarrow p_{(2)} \\ T & & T \end{array}$$

where  $F$  and  $F_{(2)}$  are the fibers of  $p$  and  $p_{(2)}$  respectively. Then

$$(1) \quad \begin{aligned} \pi_i(F) &= L_i(Z), \\ \pi_i(F_{(2)}) &= L_i(Z[1/\text{odd}]) \quad \text{for } i \geq 4. \end{aligned}$$

Here  $L_i$  denotes the Wall  $L$ -group of the rings  $Z$  and  $Z[1/\text{odd}]$ , respectively.

The map  $F \rightarrow F_2$  induces the same map on the  $L$ -groups as the ring inclusion  $Z \rightarrow Z[1/\text{odd}]$ .

(2)  $L_i(Z) \rightarrow L_i(Z[1/\text{odd}])$  is an isomorphism for  $i \neq 4k$ ; for  $i = 4k$ , it is a monomorphism with cokernel  $G$ .

Part (1) is essentially stated, along with a sketch of the proof, by L. Jones in [e]. The author also has an independent proof (for a slightly weaker result). (See Appendix, §2.)

Part (2) is based in a direct computation of  $L_*(Z[1/\text{odd}])[f]$ .

**2. Appendix.** We propose here to give some idea of how Theorem A is proved or at least of why it should be plausible. We restrict our attention to the case when  $\xi$  is a sequence of  $PL$  bundles.

**PROPOSITION.** *Let  $X^n$  be a 1-connected  $PD_{(2)}$ -space with  $\xi$ -structure,  $n \geq 5$ . Let  $f$  be the Pontrjagin-Thom map determined by the  $\xi$ -structure on  $X$ ,  $f: S^{n+k} \rightarrow T(\xi^k)$ ; and let  $g: D^{n+k+1} \rightarrow T(\xi^k)$  be an extension of  $f$ . (So, in particular the image of  $[X] \in \Omega_n^{T(\xi), 2}$  is 0 in  $\pi_n T(\xi)$ .) Then  $g$  may be deformed so as to be  $t$ -regular at 2 if and only if a certain invariant in  $L_n(Z[1/\text{odd}])$  vanishes.*

**PROOF.** We may assume that on a collar  $S^{n+k} \times I$  of  $S^{n+k}$  in  $D^{n+k+1}$ ,  $g$  is just  $f \circ$  (projection). Since  $\xi^k$  has a  $PL$  structure, we may deform  $f$  so as to be  $PL$  transverse regular, i.e.,  $f^{-1}(B_k)$  is an  $n$ -manifold  $M$  sitting inside  $R^{n+k} =$  regular neighborhood  $X$ . It turns out that there is a bundle map  $v^k(M) \rightarrow \eta^* \xi^k$  where  $\eta: X \rightarrow B_k$  is part of the  $\xi$ -structure of  $X$ . Moreover,  $M$  bounds  $W$  (by putting all of  $g$  into  $PL$  transverse position). There is a bundle map  $v^k(W) \rightarrow \xi^k$ .

Now assume  $W$  lives in the smaller copy of  $D^{n+k+1}$  got by removing the open collar on  $S^{n+k}$ , and that  $M$  lives on the ‘‘inside’’ copy of  $R^{n+k}$ . If  $M \cong X$  is a homotopy equivalence at 2, then it is easy to see that the pair  $(R \times I \cup W, R)$  is a  $PD_{(2)}$ -pair with  $\xi$ -structure (here  $R$  is essentially  $X$ ). So the problem is to make  $M \cong X$  a homotopy equivalence at 2 by doing surgery on the bundle map  $v^k(M) \rightarrow \eta^* \xi^k$  (which covers a degree-one-map of  $PD_{(2)}$ -spaces). We run into a well-defined obstruction in  $L_n(Z[1/\text{odd}])$ , which vanishes when the surgery can be completed [i]. Q.E.D.

This lemma is the key step in the author’s proof of Theorem A. In fact, the lemma by itself provides most of what is needed for a proof of Theorem 2 (which does not really need Theorem A in full generality).

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DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NEW JERSEY 08903

DEPARTMENT OF MATHEMATICS, AARHUS UNIVERSITY, AARHUS, DENMARK