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CURVATURE AND COMPLEX ANALYSIS. III

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This announcement is a sequel to Greene-Wu [**1**], [**2**]. Here we shall concentrate on Kähler manifolds of nonnegative curvature. Our first result improves Theorem 3 of [**2**], but the latter is needed in the proof of the former.

THEOREM 1. *Let M be a complete Kähler manifold with positive Ricci curvature and nonnegative sectional curvature. Let K be the canonical bundle of M and let L be a holomorphic line bundle on M such that $L \otimes K^* > 0$ (K^* denotes the dual of K; $L \otimes K^* > 0$ means that the line bundle $L \otimes K^*$ possesses a Hermitian metric of positive curvature). Then $H^p(M, \mathcal{O}(L)) = 0$ for $p \geqq 1$.*

The next theorem is the noncompact analogue of Kodaira's embedding theorem [**4**]. Its proof depends on Theorem 1 and is similar to Kodaira's proof in broad outline, but there are technical complications because of the noncompactness.

THEOREM 2. *Let M be a complete Kähler manifold with positive Ricci curvature and nonnegative sectional curvature. Then M possesses nonconstant meromorphic functions. Specifically, given any compact set $K \subseteq M$, there exists a positive integer N and a meromorphic mapping (see Remmert [**5**]) $\varphi : M \to P_N C$ such that $\varphi | K$ is a holomorphic embedding.*

In [**2**], we conjectured that every complete noncompact Kähler manifold with positive sectional curvature must be a Stein manifold. The next theorem includes the solution of this conjecture as a special case. Recall that a subset S of a Riemannian manifold is *convex* if, for any $p, q \in S$, at least one minimizing geodesic joining p and q lies in S.

THEOREM 3. *Let M be a complete Kähler manifold with positive Ricci curvature and nonnegative sectional curvature, and suppose that the canonical bundle of M is topologically trivial. Then every convex open subset of M is a Stein manifold.*

The fact that any open convex subset of such a manifold M is necessarily a Stein manifold should be compared with Theorem 7 of [**1**]; of course the

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present Theorem 3 completely supersedes Theorem 6 of [**1**]. In case M is an open subset of $\mathbf{C}^n$, we can prove that M is a Stein manifold under otherwise much weaker hypotheses.

THEOREM 4. *If M is an open subset of $\mathbf{C}^n$ which admits a complete Kähler metric G of nonnegative sectional curvature, then every convex (relative to G) open subset of M is a domain of holomorphy.*

This theorem is an improvement of Corollary (A) of Theorem 3 in [**2**]. The next result complements Theorem 5 of [**2**]. We shall use the notation of that theorem plus the following: Given a complete Kähler manifold M and a bounded subset D of M, we let r_D be the minimum of the Ricci curvature of M in $\bar{D}$.

THEOREM 5. *Let M be as in Theorem 3. Let D be a bounded pseudoconvex open subset in M and let φ be a plurisubharmonic function in D. Then, for any $f \in L^2_{(0,q)}(D, \varphi)$, $q > 0$, with $\bar{\partial} f = 0$, we can find $u \in L^2_{(0,q-1)}(D, \varphi)$ such that $\bar{\partial} u = f$ and*

$$q r_D \int_D |u|^2 e^{-\varphi} \Omega \leqq \int_D |f|^2 e^{-\varphi} \Omega.$$

The next theorem generalizes to certain Kähler manifolds of nonnegative curvature the fact that no nonzero holomorphic function on $\mathbf{C}^n$ is in L^p. The theorem is a consequence of the following result concerning Riemannian manifolds: If M is a complete noncompact Riemannian manifold of nonnegative sectional curvature and if $f \not\equiv 0$ is a nonnegative C^∞ subharmonic function, then $\int_M f\Omega = +\infty$. (Here Ω = the Riemannian volume form on M and f being subharmonic means $\Delta f \geqq 0$ everywhere on M.)

THEOREM 6. *Let M be a complete noncompact Kähler manifold with nonnegative sectional curvature. Then no nonzero holomorphic function on M is in L^p for any p satisfying $1 \leqq p < +\infty$.*

We would like to propose another conjecture. In its most conservative form, it reads: A complete Kähler manifold with positive Ricci curvature and nonnegative sectional curvature is holomorphically convex. The following theorem should be useful in resolving this conjecture. Recall that an open subset U of a complex manifold M is said to be *Runge* in M if given a holomorphic function f on U and a compact set $K \subseteq U$, there exists a holomorphic function F on M which approximates f on K arbitrarily closely. Also recall that a function on a Riemannian manifold is *convex* if its restriction to every geodesic is a convex function of one variable; a convex function is always continuous.

THEOREM 7. *Let M be a complete noncompact Kähler manifold with positive Ricci curvature and nonnegative sectional curvature. Let $\varphi : M \to \boldsymbol{R}$ be a convex function such that each sublevel set $M_c = \{p \in M : \varphi(p) < c\}$ has compact closure in M. Then M_c is Runge in M for all $c \in \boldsymbol{R}$.*

In closing, we remark that all the preceding theorems make essential use of the approximation theorem of Greene-Wu [**3**]; this fact is not surprising since that approximation theorem was proved with these applications in mind.

REFERENCES

1. R. E. Greene and H. Wu, *Curvature and complex analysis*, Bull. Amer. Math. Soc. **77** (1971), 1045–1049. MR **44** #473.
2. ———, *Curvature and complex analysis*. II, Bull. Amer. Math. Soc. **78** (1972), 866–870.
3. ———, *On the subharmonicity and plurisubharmonicity of geodesically convex functions*, Indiana Univ. Math. J. (to appear).
4. K. Kodaira, *On Kähler varieties of restricted type* (*an intrinsic characterization of algebraic varieties*), Ann. of Math. (2) **60** (1954), 28–48. MR **16**, 952.
5. R. Remmert, *Holomorphe und meromorphe Abbildungen komplexer Raüme*, Math. Ann. **133** (1957), 328–370. MR **19**, 1193.

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