

## DEGREE THEORY FOR NONCOMPACT MULTIVALUED VECTOR FIELDS

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**Introduction.** In this note we indicate the development and state the properties of a degree theory for a rather general class of multivalued mappings, the so-called ultimately compact vector fields, and then use this degree to obtain fixed point theorems. As will be seen, these results unite and extend the degree theory for single-valued ultimately compact vector fields in [13] and the degree theory for multivalued compact vector fields in ([5], [8]) and also serve to extend to multivalued mappings the fixed point theorems for single-valued mappings obtained in [1], [2], [3], [9], [10], [13], and others (see [13]) and to more general multivalued mappings the fixed point theorems in [4], [6], [8]. The detailed proofs of the results presented in this note will be published elsewhere.

1. Let  $X$  be a metrizable locally convex topological vector space. If  $D \subset X$  we denote by  $K(D)$  and  $CK(D)$  the family of closed convex, and the family of compact convex subsets of  $D$ , respectively. We also use  $\bar{D}$  (or  $\text{cl } D$ ),  $\partial D$ , and  $\text{clco } D$  to denote the closure, boundary and convex closure of  $D$ , respectively. To define what we mean when we say that the upper semicontinuous (u.s.c.) mapping  $T: D \rightarrow K(X)$  is ultimately compact, we employ a construction of a certain transfinite sequence  $\{K_\alpha\}$  utilized by Sadovsky [13] in his development of the index theory for ultimately compact single-valued vector fields. Let  $K_0 = \text{clco } T(D)$ , where  $T(A) = \bigcup_{x \in A} T(x)$  for  $A \subset D$ . Let  $\eta$  be an ordinal such that  $K_\beta$  is defined for  $\beta < \eta$ . If  $\eta$  is of the first kind we let  $K_\eta = \text{clco } T(D \cap K_{\eta-1})$ , and if  $\eta$  is of the second kind we let  $K_\eta = \bigcap_{\beta < \eta} K_\beta$ . Then  $\langle K_\alpha \rangle$  is well defined and such that  $K_\alpha \subset K_\beta$  if  $\alpha > \beta$ . Consequently, there exists an ordinal  $\gamma$  such that  $K_\beta = K_\gamma$  if  $\beta \geq \gamma$ . We define  $K = K(T, D) = K_\gamma$  and observe that  $\text{clco } T(K \cap D) = K$ . The mapping  $T$  is called *ultimately compact* if either  $K \cap D = \emptyset$  or if  $T(K \cap D)$  is relatively compact.

**DEFINITION 1.** Let  $D \subset X$  be open with  $T: \bar{D} \rightarrow K(X)$  ultimately compact and such that  $x \notin T(x)$  if  $x \in \partial D$ . If  $K(T, \bar{D}) \cap D = \emptyset$  we define  $\text{deg}(I - T, D, 0) = 0$ , and if  $K(T, \bar{D}) \cap D \neq \emptyset$  we let  $\rho$  be a retraction of  $X$  onto  $K(T, \bar{D})$  and define

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$$(1) \quad \deg(I - T, D, 0) = \deg_c(I - T\rho, \rho^{-1}(D), 0),$$

where the right-hand side of (1) means the topological degree defined in [8] for multivalued compact vector fields.

Note that the right-hand side of (1) is well defined since  $x \in D$  and  $x \in T(x)$  if and only if  $x \in \text{cl}(\rho^{-1}(D))$  and  $x \in T(\rho(x))$ , and one may show that this definition is independent of the particular retraction chosen. The combination of retractions and Leray-Schauder degree has been previously used by F. E. Browder in defining a fixed-point index, and by R. D. Nussbaum in defining the degree for single-valued  $k$ -set-contractions with  $k < 1$ . We add that if  $T$  is compact (i.e.,  $T: \bar{D} \rightarrow K(X)$  is u.s.c. and  $T(\bar{D})$  is relatively compact), then  $\deg(I - T, D, 0) = \deg_c(I - T, D, 0)$ . Furthermore, this degree has the following properties.

**THEOREM 1.** *If  $X, D,$  and  $T$  are as in Definition 1, then the degree given by (1) is such that*

- (a) *if  $\deg(I - T, D, 0) \neq 0$ , then  $T$  has a fixed point in  $D$ ;*
- (b) *if  $H: \bar{D} \times [0, 1] \rightarrow K(X)$  is u.s.c.,  $H(\bar{D} \cap K' \times [0, 1])$  is relatively compact where  $K' = K(H, \bar{D} \times [0, 1])$ , and  $x \notin H_t(x)$  for  $x \in \partial D$  and  $t \in [0, 1]$ , then  $\deg(I - H_0, D, 0) = \deg(I - H_1, D, 0)$ ;*
- (c) *if  $D = D_1 \cup D_2$ , where  $D_1$  and  $D_2$  are open and  $D_1 \cap D_2 = \emptyset$ , and  $x \notin T(x)$  for  $x \in \partial D_1 \cup \partial D_2$ , then  $\deg(I - T, D, 0) = \deg(I - T, D_1, 0) + \deg(I - T, D_2, 0)$ ;*
- (d) *if  $D$  is a symmetric neighborhood of the origin and  $T: \bar{D} \rightarrow K(X)$  is an odd ultimately compact map with  $x \notin T(x)$  for  $x \in \partial D$ , then  $\deg(I - T, D, 0)$  is odd.*

We add that even in the case of compact multivalued maps the assertion (d) is new for, unlike [5], [8], we do not require  $D$  to be convex.

2. To indicate the usefulness of the topological degree given by Definition 1 we state some examples of ultimately compact maps and some conditions under which the degree is nonzero, a condition which guarantees the existence of fixed points.

If  $\{p_\alpha | \alpha \in A\}$  is a family of seminorms which determines the topology on  $X$ ,  $\alpha \in A$  and  $\Omega \subset X$ , then we define  $\gamma_\alpha(\Omega) = \inf\{d > 0 | \Omega \text{ can be covered by a finite number of sets of } p_\alpha\text{-diameter less than } d\}$  and  $\chi_\alpha(\Omega) = \{r > 0 | \Omega \text{ can be covered by a finite number of } p_\alpha\text{-balls of radius less than } r\}$ . Letting  $C = \{f: A \rightarrow [0, \infty]\}$ , with  $C$  ordered pointwise, we define  $\gamma: 2^X \rightarrow C$  and  $\chi: 2^X \rightarrow C$  by  $\gamma(\Omega)(\alpha) = \gamma_\alpha(\Omega)$  and  $\chi(\Omega)(\alpha) = \chi_\alpha(\Omega)$  for each  $\alpha \in A$  and  $\Omega \subset X$ . Then  $\gamma$  and  $\chi$  are measures of noncompactness which possess the usual properties (see [14] for  $\chi$ ) and we let  $\Phi$  denote either  $\chi$  or  $\gamma$ . A u.s.c. map  $T: D \rightarrow CK(X)$  is called  $\Phi$ -condensing if  $\Phi(T(A)) \not\subseteq \Phi(A)$  when  $A \subset D$  and  $\Phi(A) \neq 0$  and, if  $k \in R$ ,  $T$  is called a  $k$ - $\Phi$ -con-

traction if  $T(D)$  is bounded and  $\Phi(T(A)) \not\asymp k\Phi(A)$  when  $A \subset D$ . Recall that when  $X$  is a Banach space, then  $T: D \rightarrow CK(X)$  is called *contractive* (*nonexpansive*) if there exist  $\alpha \in (0, 1)$  ( $\alpha = 1$ ) such that

$$(1) \quad d^*(T(x), T(y)) \leq \alpha d(x, y) \quad \text{for } x, y \in D,$$

where  $d^*$  is the Hausdorff metric on  $CK(X)$  derived from  $d = \|\cdot\|$ . Finally, following [7], we say that a u.s.c. map  $T: D \rightarrow K(X)$  is *generalized condensing* if for each  $Q \subset D$  such that  $T(Q) \subset Q$  and  $Q \setminus \text{clco } T(Q)$  is relatively compact, the set  $\bar{Q}$  is compact.

It is clear that every generalized condensing map  $T: D \rightarrow D$  is ultimately compact if  $D$  is closed and convex. Further, if  $X$  is also complete (i.e., a Fréchet space), then every  $k - \phi$ -contraction  $T: \bar{D} \rightarrow CK(X)$  with  $0 < k < 1$  is  $\phi$ -condensing, and every  $\phi$ -condensing map is ultimately compact.

**THEOREM 2.** *Let  $D \subset X$  be convex and open and let  $T: \bar{D} \rightarrow K(\bar{D})$  be ultimately compact with  $K(T, \bar{D}) \neq \emptyset$  and  $x \notin T(x)$  for  $x \in \partial D$ . Then  $\deg(I - T, D, 0) = 1$ , and so  $T$  has a fixed point.*

It is not hard to show that if  $T$  in Theorem 2 is generalized condensing (and, in particular, if  $T$  is  $\phi$ -condensing and  $X$  also complete), then  $K(T, \bar{D}) \neq \emptyset$  and so Theorem 2 is valid for these classes of maps without the explicit assumption that  $K(T, \bar{D}) \neq \emptyset$ .

**THEOREM 3.** *Let  $X$  be a Fréchet space,  $D$  a neighborhood of 0, and  $T: \bar{D} \rightarrow CK(X)$  a  $\phi$ -condensing map such that*

$$(2) \quad \{\lambda x\} \cap T(x) = \emptyset \quad \text{for } x \in \partial D \text{ and } \lambda \geq 1.$$

*Then  $T$  has a fixed point.*

Using Theorem 3 one proves the following general fixed point theorem for  $1 - \phi$ -contractions.

**THEOREM 4.** *Let  $X$  and  $D$  be as in Theorem 3 and let  $T: \bar{D} \rightarrow K(X)$  be a  $1 - \phi$ -contraction. Suppose further that if there is a sequence  $\{x_n\} \subset \bar{D}$  with corresponding  $y_n \in T(x_n)$  for each  $n$  such that  $x_n - y_n \rightarrow 0$  as  $n \rightarrow \infty$ , then there exists  $x \in \bar{D}$  with  $x \in T(x)$ . If  $T$  satisfies (2), then  $T$  has a fixed point in  $\bar{D}$ .*

For  $X$  a Hausdorff l.c.t.v.s. and  $T$  single-valued, Theorem 2 was proved by Sadovskiy [13]. For  $X$  a Banach space and  $T$  single-valued and  $\gamma$ -condensing, Theorem 3 was deduced in [10] from the index theory for  $\gamma$ -condensing maps developed in [9]; for multivalued  $\chi$ -condensing maps, Theorem 3 includes the result of [6], while for multivalued compact maps with  $D = B(0, r)$ , Theorem 3 was proved in [4] for  $X$  a Banach space and in [8] for  $D$  a neighborhood of 0 in a Hausdorff l.c.t.v.s. In case  $X$  is Banach

and  $T$  single-valued, Theorem 4 reduces to Theorem 1 in Petryshyn [11].

Our next result extends to condensing maps  $T: \bar{D} \rightarrow K(X)$  and to symmetric but not necessarily convex sets  $D$  the validity of the antipodes theorem established in [4], [8] for compact multivalued maps and in [13] for single-valued condensing maps.

**THEOREM 5.** *Let  $X$  be a Fréchet space,  $D$  a symmetric neighborhood of  $0 \in X$ , and  $T: \bar{D} \rightarrow CK(X)$   $\phi$ -condensing. Assume also that*

$$(3) \quad \{x - T(x)\} \cap \lambda\{-x - T(-x)\} = \emptyset \quad \text{for } x \in \partial D \text{ and } \lambda \in [0, 1].$$

*Then  $\deg(I - T, D, 0)$  is an odd integer.*

Since a contraction  $T: X \rightarrow CK(X)$  is  $\alpha$ -ball-contractive,  $0 < \alpha < 1$ , on each bounded set in  $X$ , an immediate consequence of Theorems 3 and 5 is the following corollary.

**COROLLARY 1.** *Let  $X$  be a Banach space,  $D$  a bounded neighborhood of  $0 \in X$ ,  $S: X \rightarrow CK(X)$  contractive, and  $C: \bar{D} \rightarrow CK(X)$  compact. If  $T = S + C: \bar{D} \rightarrow CK(X)$  satisfies either (2) or (3) on  $\partial D$ , then  $T$  has a fixed point in  $T$ .*

If in Corollary 1 the map  $S$  is defined only on  $\bar{D}$ , then the conclusion still holds provided that either  $S$  is single-valued, or the constant  $\alpha$  in (1) is  $< \frac{1}{2}$ , or  $X$  is a Hilbert space and  $D = B(0, r)$ .

We add in passing that if the Banach space is assumed to have the so-called Opial property,  $D$  is weakly compact, and  $C: \bar{D} \rightarrow CK(X)$  is completely continuous, then Corollary 1 also holds for  $S: X \rightarrow CK(X)$  nonexpansive.

We end our note with the following mapping theorem which extends the corresponding result of Ma [8] for multivalued compact maps.

**THEOREM 6.** *Let  $X$  be a Fréchet space,  $D \subset X$  an open set, and  $T: D \rightarrow K(X)$   $k - \phi$ -contractive with  $k < 1$ . If  $T$  is a boundary map in the sense of Ma [8], then  $(I - T)(D)$  is open.*

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