THE TANGENT SPACE TO A \( C^k \) MANIFOLD

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Communicated by S. S. Chern, December 27, 1972

The algebraic tangent space to a finite dimensional \( C^k \) manifold 
\((1 \leq k \leq \infty)\) is the vector space of linear derivations on \( C^k_p \), the ring of germs of real \( C^k \) functions at \( p \). In this note we give a short proof that, for \( k < \infty \), the algebraic tangent space is infinite dimensional.

This result is well known but we believe the proof presented here is the easiest. An apparently incorrect proof was given by Papy in [3]. A proof for the case \( k = 1 \) was given by Osborn in [2]. A complete solution is given by Newns and Walker in [1].

Let \( I \) be the maximal ideal of \( C^k_p \). One sees (as in, for example, [4, p. 13]) that the algebraic tangent space is canonically isomorphic to \((I/I^2)^*\). We shall show \( I/I^2 \) is infinite dimensional.

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LEMMA. If \( f \in I^2 \) then \( o(f) > k + 1 \) or is an integer.

PROOF. Note that if \( g \in I \) we can, using Taylor's theorem, write \( g = \sum a_j x^j \) where \( a_j \) are real and \( r \) is the germ of a continuous function at \( p \). But it follows that we can write \( f = \sum a_j x^j + x^{k+1}r \). If all the \( a_j \) vanish \( o(f) \geq k + 1 \). If not, \( o(f) = \min \{ j : a_j \neq 0 \} \).

We now show \( I/I^2 \) is infinite dimensional by proving \( \{ |x^\sigma : k < \sigma < k + 1 \} \) is a linearly independent collection. For, if not, we have \( \sum c_i |x|^{\sigma_i} = g \) where no \( c_i \) vanishes, the \( \sigma_i \) are distinct, and \( g \in I^2 \). But then \( o(g) = \min \{ \sigma_i \} \), which is a noninteger less than \( k + 1 \) contradicting the lemma.

REFERENCES


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AMS (MOS) subject classifications (1970). Primary 58A05; Secondary 16A72.

Key words and phrases. Tangent space, manifold of class \( C^k \), derivation.