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BOOK REVIEWS

Free rings and their relations, by P. M. Cohn. Academic Press, London & New York, 1971. xvi + 346 pp. $22.00

Free associative algebras, i.e., polynomial rings in noncommuting variables, are not mentioned in most of the standard texts in ring theory; Cohn's book is the first comprehensive treatment of this subject. The book is up-to-date, very well written and essentially self-contained.

The lack of reference to free rings in previous works is understandable; these rings stand somewhat apart from the traditional branches of ring theory. The lack of finiteness conditions in free algebras (in fact they are automatically infinite dimensional) dissociates them from the classical theory of noncommutative algebras which were always assumed to be finite dimensional vector spaces over the base field. On the other hand, the noncommutativity of free rings separates them from the other main branch of ring theory: algebraic number fields and their generalizations.

Recently, a new way of approaching problems in algebraic geometry has provided a geometric insight into many notions of ring theory. But the commutativity restrictions cannot be easily circumvented. Noncommutative algebraic geometry is almost nonexistent, and before such a theory could be done one would have to develop a theory of algebraic equations in noncommuting indeterminates. At present, not enough is known about the structure of free rings, and we are far from being able to handle some of the most elementary problems that occur in this field. Most of the questions one asks are simple translations from the commutative case and from the theory of free groups. Unfortunately, the methods cannot be successfully adapted in most cases, and the translated "theorems" are not even true.

The systematic development of the subject has been done mainly by P. M. Cohn during the last decade. Many of the theorems in this book are taken directly from the author's papers and from G. M. Bergman's thesis (Harvard 1967), in which some outstanding open problems were solved and existing results were simplified and generalized. Though this book deals with a very specialized subject within ring theory, the author has been careful, at every stage, to establish the connection with the traditional branches by showing how many of the standard notions can be generalized.

The book does not contain a complete account of related topics (in fact, a second volume is mentioned). For example the construction of firs and free products is not included. Aside from some minor misprints I have not found any major errors in the text.

I highly recommend the book to researchers in the field and especially to graduate students; it contains references to a multitude of open problems, the solution of which would considerably advance the subject.

A detailed description of the contents follows.

Some acquaintance with elementary notions of ring theory is assumed. The general background from lattice theory, categorical and homological algebra and some special notions such as Morita equivalence, Ore rings and rings of fractions are included in the **Appendix** and **Chapter 0.**

If F is a field, the *free associative algebra over* F on a set X is denoted by $F\langle X\rangle$ and defined by the following universal property: $F\langle X\rangle$ is an F-algebra with a mapping $i:X \to F\langle X\rangle$ such that any other mapping $\theta:X \to A$, where A is an F-algebra, can be factored uniquely by i.

More constructively, let $X = \{x_j\}_{j\in J}$. The free semigroup S_X on X consists of all formal products $x_I = x_{j_1}x_{j_2}\cdots x_{j_n}$ where $I = (j_1, \ldots, j_n)$ runs over all finite sequences of suffixes (including the empty sequence to obtain 1). $F\langle X\rangle$ can then be described as the semigroup algebra of S_X over F. Hence its elements are uniquely expressible in the form: $f = \sum_I x_I a_I$, where the a_I's $\in F$ and are all zero except for a finite number.

Chapter 1 starts by describing a class of rings that generalizes the notion of principal ideal domains, namely that of "firs". A ring R is said to be a *right fir* if all its one-sided right ideals are free right modules of unique rank. A less restrictive notion is that of a *right n-fir*, i.e., a ring such that all its right ideals generated by n or less elements are free of unique rank. The latter notion is left-right symmetric and some equivalent characterizations of n-firs are given in terms of trivializing certain relations of dependence between elements of the ring by means of invertible matrices.

Since a 1-fir is just an integral domain, and hence too general for most purposes, the next important case considered is that of 2-firs which are then described as integral domains in which any two principal right (or left) ideals with nonzero intersection have a sum that is principal.

One of the most useful results is the following description: An integral domain R is a 2-fir if and only if for every $c \neq 0$ in R, the principal right ideals containing c form a sublattice $L(cR, R)$ of the lattice of all right ideals of R; the lattice $L(cR, R)$ is, in this case, what is called *modular*.

Chapter 2 is devoted to rings with a "weak algorithm" which is a generalization of the Euclidean algorithm and can be used much in the same way.

A *filtration* v for a ring R is a map from R to the natural numbers together with $-\infty$, satisfying:

V.1. $v(x) \geqq 0$ for all $x \neq 0$, $v(0) = -\infty$,

V.2. $v(x - y) \leqq \max\{v(x), v(y)\}$, V.3. $v(xy) \leqq v(x) + v(y)$,

V.4. $v(1) = 0$.

A ring R with a filtration v is said to satisfy the *n-term weak algorithm* with respect to v if given a family $a_1, \ldots, a_m$ $(m \leqq n)$ of elements of R such that: (i) $v(a_1) \leqq \cdots \leqq v(a_m)$ and (ii) there are elements $b_1, \ldots, b_m$ of R satisfying $v(\sum_j a_j b_j) < \max_j\{v(a_j) + v(b_j)\}$; then there exist elements $c_1, \ldots, c_{i-1}$ of R satisfying $v(a_i - \sum_{j=1}^{i-1} a_j c_j) < v(a_i)$ for some i and $v(a_j) + v(c_j) \leqq v(a)$. If R satisfies the n-term weak algorithm for all n, we say that R satisfies the *weak algorithm for v*.

Even though this defines a ring with a right n-term weak algorithm it is shown that the notion is left-right symmetric.

If there exists a filtration v for which the ring R satisfies the 1-term weak algorithm, it follows that R is a (noncommutative) integral domain. Furthermore, if R satisfies the n-term weak algorithm, then R satisfies the m-term weak algorithm for all $m \leqq n$. Consequently, most of the rings under consideration are assumed to be integral domains.

The weak algorithm is then used to completely characterize free algebras. The following result is proved:

Any free associative algebra $F\langle X\rangle$ on a set $X = \{x_j\}_{j\in J}$ satisfies the weak algorithm with respect to the filtration obtained by assigning arbitrary positive degree to the elements of X, say $v(x_j) \in N^+$, $j \in J$, and setting for every $f = \sum_I x_I a_I$ in $F\langle X\rangle$:

$$v(f) = \max_{I=(i_1,\ldots,i_n);\, a_I \neq 0} \{v(x_{i_1}) + \cdots + v(x_{i_n})\}.$$

Conversely, any algebra R over a commutative field F that satisfies the weak algorithm with respect to a filtration v for which F constitutes the elements of R of zero degree, is the free associative F-algebra on a set X, and the valuation v can be described as above. As a corollary it is proved that every free associative algebra over a commutative field is a 2-sided fir.

The next result is Bergman's classification of all rings with a weak algorithm.

There is also an inverse weak algorithm that is used to describe noncommuting power series rings, and finally, a transfinite weak algorithm that is the main tool in the construction of an example of a right fir which is not a left fir.

Chapter 3 deals with the notion of unique factorization in noncommutative rings.

Two elements a and a' of a ring R are called (right) *similar* if the right modules R/aR and $R/a'R$ are isomorphic. This notion is left-right symmetric for nonzero divisors and in commutative rings it reduces to the notion of associated elements.

Two factorizations of a given element c of R, say $c = a_1 \cdots a_r$ and $c = b_1 \cdots b_s$ are said to be *isomorphic* if $r = s$ and there exists a permutation $i \to i'$ of $1, \ldots, r$ such that b_i is similar to $a_{i'}$.

A noncommutative *unique factorization domain* (U.F.D. for short) is then defined as an atomic integral domain such that any two atomic factorizations of a given element are isomorphic. This definition reduces to the usual one if R is commutative.

It is proved that every atomic 2-fir is a U.F.D., and since every filtered ring with 2-term weak algorithm is an atomic 2-fir, any free associative algebra is a U.F.D.

The unique factorization in noncommutative domains is more complex than in the commutative case because similar elements are not necessarily associated and a reordering of the factors will change the product. For example in the free algebra $F\langle x, y\rangle$ the element $xyx + x$ has two atomic factorizations: $(xy + 1)x = x(yx + 1)$ where $xy + 1$ is similar to $yx + 1$.

To obtain sharper results about factorizations **Chapter 4** discusses rings in which the lattice $L(cR, R)$ of right factors of a nonzero element c is not only modular but also *distributive*, i.e., it does not contain a 5-element sublattice of length 2.

If R is a commutative domain for which the lattice of factors of an element is modular, then there is an induced multiplication in this lattice, compatible with the ordering, that makes the lattice distributive. In the noncommutative case, the distributivity of $L(cR, R)$ is not automatic, and hence this property singles out a special kind of 2-firs, rings R for which $L(cR, R)$ is distributive for all nonzero $c \in R$. Rings having this property are said to have a *distributive factor lattice*.

In particular, every free associative algebra has a distributive factor lattice. The main consequence of this result is that the factorization of elements can be described in greater detail.

Another important result is that every finite distributive lattice can be realized as the lattice of factors of an element in a free associative algebra. More precisely, if $A_n = F\langle x_1, \ldots, x_n\rangle$ and P is any partially ordered set in n elements, there exists an element $c \in A_n$ such that the partially ordered set P_c of lower segments of the lattice $L(cA_n, A_n)$ is isomorphic to P. Further, all finite partially ordered sets can be realized in this way in A_2 since A_n, and in fact any free associative algebra of countable rank, can be embedded in A_2 in such a way that factorizations are preserved.

The situation here is completely different from the commutative U.F.D. in which the only finite partially ordered sets that can be so realized are the disjoint union of finite chains.

In the commutative case, the factorization theory of principal ideal domains (P.I.D.) can be extended to matrices over such rings and this leads to the structure theory of finitely generated modules over P.I.D. This theory can be partly generalized to firs (and in particular to free algebras), leading to factorization theorems for matrices over firs.

Chapter 5 starts by investigating a class of modules over firs. If M is a right module over a fir R, with a presentation $0 \to R^m \to R^n \to M \to 0$, one defines the *characteristic* of M to be $\chi(M) = n - m$; this is a well-defined notion because R is a fir. Now, M is said to be a *torsion module* if (i) $\chi(M) = 0$ and (ii) for all submodules M' of M, $\chi(M') \geqq 0$. This notion reduces to the usual one if R is commutative.

The category of torsion modules over a fir is an abelian category which is Artinian and Noetherian, i.e., a torsion module over a fir satisfies both chain conditions on torsion submodules.

The factorization of elements is then carried over to matrices and it turns out that any regular square matrix over a 2-sided fir can be written as a product of atoms, and any two such atomic factorizations are isomorphic.

This chapter also contains the intersection theorem for twosided ideals: In a twosided fir the intersection of the powers of any proper twosided ideal is zero.

Chapter 6 investigates certain subrings of firs and semifirs. The center of a fir can take only a very special form: a commutative ring can be the center of a 2-fir if and only if it is integrally closed; further, it can be the center of a (noncommutative) P.I.D. if and only if it is a Krull domain. If R is a fir which if not a P.I.D. its center must be a field.

The chapter then concentrates on some subalgebras of free algebras. The main open problem here is to decide which subalgebras of a free associative algebra are themselves free. For example, P. M. Cohn proved that a subalgebra of the polynomial ring $F[x]$ in one variable is free iff it is integrally closed; but if the free algebra has more than one generator no reasonable criterion is known for a subalgebra to be again free.

This chapter ends with a fundamental theorem due to Bergman: In any free associative algebra $F\langle X\rangle$ over a commutative field F the centralizer of a nonscalar element of $F\langle X\rangle$ is a polynomial ring in one variable over F.

Chapter 7 presents a way of embedding rings in fields and in general of constructing homomorphisms of rings into fields. For a survey which includes most known methods of embedding rings into fields the reader is referred to: P. M. Cohn, *Rings of fractions*, Amer. Math. Monthly **78** (1971), 596–615.

The material of this chapter is based on P. M. Cohn, *Universal skew fields of fractions*, Symposia Math. **8** (1972), and the basic idea involved is that of inverting matrices instead of elements. One defines a matrix ideal to be a certain collection of square matrices over a ring R, which is closed under two operations. It is possible to develop a theory of prime matrix ideals having essentially the properties of prime ideals of a ring. The main theorem states that given a prime matrix ideal $\mathscr{P}$ of a ring R there exists a field K and a homomorphism $R \to K$ such that $\mathscr{P}$ is precisely the class of matrices mapped to singular matrices under the homomorphism $R \to K$. The field K is obtained by "localizing" R at the "multiplicatively closed" set Σ consisting of all square matrices in the complement of $\mathscr{P}$.

This theorem is used to give a necessary and sufficient criterion for the embeddability of a ring into a skew field.

Chapter 8 is devoted to special properties of (noncommutative) P.I.D., for example, the diagonal reduction of matrices. In particular, every finitely generated module M over a P.I.D. R is a direct sum of cyclic modules:

$$M = R/e_1R \oplus \cdots \oplus R/e_rR \oplus {}^{m-r}R$$

where e_i is a total divisor of e_{i+1} for $i = 1, \ldots, r-1$ and this condition determines the e_i's up to similarity. (That e is a *total divisor* of e' means that there exists an element c of R such that $cR = Rc$ and e divides c and c divides e'.)

The results are applied to characterize the invariant elements of the skew polynomial ring $R = k[t; S, D]$ with automorphism S and S-derivation D, and also to investigate certain algebraic extensions of skew fields.

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Stabilité Structurelle et Morphogénèse, Essai d'une Théorie Générale des Modèles, by René Thom. Benjamin, New York, 1971. 384 pp. $20.

René Thom has written a provocative book. It contains much of interest to mathematicians and has already had a significant impact upon mathematics, but *Stabilité Structurelle et Morphogénèse* is not a work of mathematics. Because Thom is a mathematician, it is tempting to apply mathematical standards to the work. This is certainly a mistake since Thom has made no pretense of having tried to meet these standards. He even ends the book with a plea for the freedom to write vaguely and intuitively without being ostracized by the mathematical community for doing so. Instead of insisting that Thom's style conform to prevailing norms, we should applaud him for sharing his wonderful imagination with us.

The book touches upon an enormous spectrum of material from developmental biology to optics to linguistics as well as mathematics. I can