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CLASSIFYING RELATIVE EQUILIBRIA. I

BY JULIAN I. PALMORE

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Introduction. We announce several theorems which place us nearer the goal of classifying relative equilibria in the planar n-body problem. These theorems answer some of the questions on the nature of relative equilibria which were raised recently by S. Smale [3], [4]. We refer to these papers for definitions left unspecified here. It is a pleasure to acknowledge the encouragement of S. Smale in these pursuits.

1. **Relative equilibria defined.** We study a real analytic function $\tilde{V}_m$ on a real analytic manifold X_m where $n \geqq 3$ and $m = (m_1, \ldots, m_n) \in R_+^n$ are fixed. For each n, m, X_m is homeomorphic to $P_{n-2}(C) - \tilde{\Delta}_{n-2}$, an open manifold of dimension $2n - 4$. $\tilde{\Delta}_{n-2}$ is the union of $n(n-1)/2$ codimension 1 complex projective subspaces. $\tilde{V}_m$ is the potential function which is induced on X_m by V_m, the potential function of the planar n-body problem.

The classical definition of relative equilibria [5, p. 286] and the modern one [4, p. 47] are equivalent. For our purposes we may consider a class of relative equilibria to be a critical point $x \in X_m$ of $\tilde{V}_m$. It is known that $\tilde{V}_m$ is a proper map and from [2] that the critical points of $\tilde{V}_m$ are bounded away from the fat diagonal $\tilde{\Delta}_{n-2}$. Therefore, provided $\tilde{V}_m$ is nondegenerate, we can apply Morse theory in order to count critical points.

As a first step we need to know the homology of the manifold $P_{n-2}(C) - \tilde{\Delta}_{n-2}$ before we attempt to apply the Morse inequalities directly. The answer as given by Theorem 1 is a recurrence relation for the

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integral singular homology of $P_{n-2}(C) - \tilde{\Delta}_{n-2}$. As a step toward asserting the nondegeneracy of $\tilde{V}_m$ we give Theorem 2 which places on the index of any critical point of $\tilde{V}_m$ (degenerate or not) a lower bound. Theorem 3 gives the index of distinguished critical points (the Moulton classes). As a corollary, it follows that the lower bound given by Theorem 2 is the greatest lower bound on the index.

Finally, we apply these results to the cases of three masses and four equal masses. In the latter we bring to light in Theorem 5 the unexpected existence of many new classes of relative equilibria.

2. **Main theorems.** In this paragraph let χ_n denote the Euler characteristic of $P_{n-2}(C) - \tilde{\Delta}_{n-2}$ and $\beta_i = \operatorname{rank}(H_i(P_{n-2}(C) - \tilde{\Delta}_{n-2}))$, $0 \leqq i \leqq 2n - 4$ where H_* is the integral singular homology.

THEOREM 1. *For any $n \geqq 3$ and any i, $0 \leqq i \leqq n - 2$,*

$$H_i(P_{n-2}(C) - \tilde{\Delta}_{n-2}) \cong H_i(P_{n-3}(C) - \tilde{\Delta}_{n-3}) \oplus \bigoplus(n-1)$$

copies $H_{i-1}(P_{n-3}(C) - \tilde{\Delta}_{n-3})$. For $i > n - 2$, $H_i(P_{n-2}(C) - \tilde{\Delta}_{n-2}) = 0$. The homology H_* is torsion free over Z.

COROLLARY 1.1. *$H_*(X_m)$ is independent of $m \in R^n_+$.*

COROLLARY 1.2. *$\beta_i = (-1)^i \sum_{k=0}^{i} S_n^{n-k}$ for all i, $0 \leqq i \leqq n - 2$, and $n \geqq 3$.*

Here S_p^q is a Stirling number of the first kind; i.e., $(-1)^{p-q} S_p^q$ is the number of permutations of p elements with exactly q cycles.

COROLLARY 1.3. *$\chi_n = (-1)^n (n-2)!$ for any $n \geqq 3$.*

COROLLARY 1.4. *$\sum_{i=0}^{n-2} \beta_i = n!/2$ for any $n \geqq 3$.*

REMARK. Corollary 1.4 gives the minimum number of critical points which exist on X_m according to Morse theory for any C^2 nondegenerate proper real valued function. It is interesting to note that the number of Moulton classes of relative equilibria equals $n!/2$ for $n \geqq 2$ and any $m \in R^n_+$ [**1**], [**4**].

Let $x \in X_m$ be a critical point of $\tilde{V}_m$. The index of $\tilde{V}_m$ at x is the maximal dimension of the subspace of $T_x X_m$ on which the hessian of $\tilde{V}_m$, $D^2 \tilde{V}_m(x)$, is negative definite. In this definition we allow x to be a degenerate critical point. We assign to each such x a positive integer or zero (denoted by $\operatorname{ind}(x)$) which equals the index of $\tilde{V}_m$ at x.

The homology of X_m is trivial in dimensions greater than $n - 2$. We want to know whether or not the index of a critical point is bounded in an analogous way. The next two theorems provide the best answer.

THEOREM 2. *For any critical point $x \in X_m$ of $\tilde{V}_m$, $\operatorname{ind}(x) \geqq n - 2$.*

Let $P_{n-2}(R) - \tilde{\Delta}_{n-2} \subset P_{n-2}(C) - \tilde{\Delta}_{n-2}$ and let $Y_m \subset X_m$ be the submanifold which corresponds to $P_{n-2}(R) - \tilde{\Delta}_{n-2}$. A critical point $x \in Y_m \subset X_m$ of $\tilde{V}_m$ is called a Moulton class of relative equilibria. Each member of a Moulton class has its masses $m_1, \ldots, m_n$ positioned on a line E^1 in the plane E^2 which passes through the origin.

THEOREM 3. *Let $x \in Y_m \subset X_m$ be a critical point of $\tilde{V}_m$ for any $n \geqq 2$ and any $m \in R^n_+$. Then x is a nondegenerate critical point of $\tilde{V}_m$ and* $\operatorname{ind}(x) = n - 2$.

COROLLARY 3.1. *The greatest lower bound on the indexes of critical points of $\tilde{V}_m$ for any $n \geqq 3$ and any $m \in R^n_+$ equals $n - 2$.*

INDICATION OF PROOF OF THEOREM 2. The hessian of $\tilde{V}_m$ at a critical point $x \in X_m$ is invariantly defined. The maximal dimension of the subspace of T_xX_m on which $D^2\tilde{V}_m(x)$ is negative definite is equal to the number of negative eigenvalues of the matrix of $D^2\tilde{V}_m(x)$ with respect to any basis of T_xX_m. We introduce a basis of T_xX_m to show the existence of one negative eigenvalue and proceed by recurrence to show the existence of precisely $n - 2$ negative eigenvalues.

INDICATION OF PROOF OF THEOREM 3. We establish an estimate on the location of the Moulton points. The estimate bounds the distance between two adjacent masses in the collinear configuration and it is independent of the number of masses. We show that for each Moulton point there is a basis of T_xX_m which splits the tangent space into two invariant subspaces of dimension $n - 2$ and that on one of these the hessian is positive definite. By Theorem 2 we are done.

3. **Classifying relative equilibria.** For any three positive masses the only relative equilibria are those which correspond to Lagrange's equilateral triangle solutions and Euler's three collinear configurations [**5**, pp. 275–277]. These relative equilibria vary continuously with the masses. It only remains to show that the five classes of relative equilibria are nondegenerate.

THEOREM 4. *For any three positive masses the index of a Lagrange class equals* 2 *and the index of an Euler class equals* 1.

COROLLARY 4.1. *$\tilde{V}_m$ is a Morse function for $n = 3$.*

Wintner [**5**, p. 277] suggests that the largest contribution to the classes of relative equilibria is due to the collinear configurations (Moulton classes). This is true for $n = 3$ but appears to be false for $n \geqq 4$. It follows easily from Theorems 1 and 3 that whenever $\tilde{V}_m$ is nondegenerate the minimum number of critical points equals $(3n - 4)(n - 1)!/2$ and the ratio of the number of classes $x \in X_m - Y_m$ to the number of Moulton classes ($= n!/2$) exceeds 1 when $n > 4$. This ratio equals 1 when $n = 4$ but as we see below in this case the suggestion fails spectacularly.

To illustrate the application of these theorems we examine the case of four equal masses. Besides the 12 Moulton classes there is an equilateral configuration of three masses with a fourth mass at the origin ($=$ center of mass) and a configuration of four masses on the vertices of a square whose center lies at the origin. The equilateral configuration represents 8 classes and the square configuration represents 6 classes. Each class in an equilateral configuration has index equal to 2 and each class in a square configuration has index equal to 4. All of these classes are nondegenerate.

The Betti numbers of X_m which enter into the right hand side of the Morse inequalities are $\beta_2 = 6$, $\beta_3 = 5$, and $\beta_4 = 1$. Assuming that $\tilde{V}_m$ is nondegenerate in this case the Morse inequalities at once yield that the number of critical points with index equal to 3 must equal or exceed 24. An equilibrium configuration of four equal masses which can be counted as 24 classes (where all of these classes have the same index) has at most one axis of symmetry. This property suggests the configurations to look for. If $\tilde{V}_m$ is not a Morse function then other, degenerate, critical points must exist.

The following theorem summarizes the new results so far obtained.

THEOREM 5. *Given four equal masses there exist* 120 *classes of relative equilibria such that each member of* 24 *of the classes has the shape of an isosceles triangle with one mass at each vertex and with a fourth mass in the interior on the axis of symmetry. The members of each class can be placed into one of five geometrically distinct triangular configurations each of which corresponds to* 24 *classes of relative equilibria under permutation of the masses.*

For $n > 4$ whenever $\tilde{V}_m$ is nondegenerate classes of relative equilibria exist having indexes in the range $n - 2 \leqq \mathrm{ind}(x) \leqq 2n - 4$. The only equilibria known to exist for any $n > 4$ and any $m \in R^n_+$ are the Moulton classes and the maxima of $\tilde{V}_m$. It is unknown whether there always exist classes of relative equilibria with indexes in the range $n - 2 < \mathrm{ind}(x) < 2n - 4$. In particular the major question remains open: Is the number of critical points of $\tilde{V}_m$ always finite?

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA 94720

Current address: Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139