GLOBAL DEFINABILITY THEORY IN $L_{\omega_1\omega}$

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Communicated by A. H. Lachlan, February 23, 1973

Introduction. Results in definability theory which are about a fixed structure are called “local” by Reyes [R]. An example is Scott’s definability theorem [Sc]. In contrast, “global” results are about the class of models of a sentence (theory); an example is Svenonius’ theorem [Sv]. Note that the straight analogue for $L_{\omega_1\omega}$ of Svenonius’ theorem, if true, would be a global generalization of Scott’s theorem, i.e., the latter would be obtained by applying the former to the Scott sentence of the given structure. Although this generalization is false, Motohashi [Mo] has found a totally satisfactory global generalization of Scott’s theorem (his result is explained below).

We give two distinct global generalizations of a local weak-definability theorem by Kueker [Ku 1] and Reyes [R] (Theorems 1 and 2 and Corollary (A)) and one for Kueker’s local theorem in [Ku 1] on structures with only countably many automorphisms (Theorem 3 and Corollary (E)). In Theorems 2 and 3, we utilize Motohashi’s work. Theorem 4 is related to [Ku 2].

1. Results. $L$ denotes a fixed countable language, $L_{\omega_1\omega}$ the finite-quantifier logic with countable conjunctions and disjunctions based on $L$. $P$ is an additional predicate symbol, $L_{\omega_1\omega}(P)$ is the corresponding extension of $L_{\omega_1\omega}$. $\mathfrak{A}$ and $(\mathfrak{A}, P)$ denote structures for $L_{\omega_1\omega}$ and $L_{\omega_1\omega}(P)$, resp. Following [Ku 1], we write $M_\sigma(\mathfrak{A})$ for $\{P: (\mathfrak{A}, P) \models \sigma\}$ and $M(\mathfrak{A}, P)$ for $\{Q: (\mathfrak{A}, Q) \text{ is isomorphic to } (\mathfrak{A}, P)\}$. $|X|$ is the cardinality of $X$.

**Theorem 1.** For any sentence $\sigma$ in $L_{\omega_1\omega}(P)$, (i) $\iff$ (ii).

(i) For every countable $\mathfrak{A}$, $|M_\sigma(\mathfrak{A})| \leq \aleph_0$ (or, equivalently, $< 2^{\aleph_0}$).

(ii) For some formulas $\varphi_n(\bar{x}, \bar{u})$ ($n < \omega$) of $L_{\omega_1\omega}$,

$$\sigma \models \bigvee_{n < \omega} \exists \bar{u}^n \forall \bar{x} [\exists \bar{x} [P \bar{x} \leftrightarrow \varphi_n(\bar{x}, \bar{u}^n)]]$$

Theorem 1 is a direct analogue of the weak-definability theorem for finitary logic of Chang [C] and the author [Ma 1], as improved by Reyes [R] for countable structures. In fact, our proof gives the result for all admissible fragments of $L_{\omega_1\omega}$ (with the whole formula after “$\models$” in (ii) being in the fragment). A similar remark applies for our subsequent

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1 Research supported by the National Research Council of Canada.
GLOBAL DEFINABILITY THEORY IN $L_{\omega_1\omega}$

results. Taking $\sigma$ to be the Scott sentence of $(\mathcal{U}, P)$, we obtain

**Corollary (A)** (KUEKER [Ku 1], REYES [R]). $|M(\mathcal{U}, P)| \leq \aleph_0$ \iff \(|M(\mathcal{U}, P)| < 2^{\aleph_0}\) \iff for some finitely many elements $a$ of $A$, $P$ is definable in $(\mathcal{U}, a)$ by an $L_{\omega_1\omega}$-formula with the parameters $a$.

Our next two theorems utilize work of Motohashi [Mo]. Let $X$, $Y$ be disjoint infinite sets of variables. $x, x_i, \ldots$ denote variables from $X$; $y, y_i, \ldots$ from $Y$; $\bar{x}, \bar{x}^i, \ldots$ vectors of $x$'s, similarly for $\bar{y}, \bar{y}^i$.

**Definition (Motohashi)**. A formula $\theta$ in $L_{\omega_1\omega}(P)$ is called a Motohashi formula ($M$-formula) if every atomic subformula of $\theta$ is of the form either $\forall(x)$ or $\exists(x)$ with $\forall(-)$ in $L_{\omega_1\omega}$ or else $P_a$.

The following are easily seen.

**Proposition (B)** ([Mo]). A finitary $M$-formula $\theta(\bar{x})$ is logically equivalent to a finitary formula of the form $\bigwedge_{i < \omega} [\sigma_i \rightarrow \varphi_i(\bar{x})]$, $\sigma_i$ sentences in $L_{\omega_1\omega}(P)$, $\varphi_i(\bar{x})$ in $L_{\omega_1\omega}$.

(C) For given countable $(\mathcal{U}, P)$, $\theta(\bar{x}, \bar{y})$ an $M$-formula, $a^0$ elements of $A$, $\theta(\bar{x}, \bar{a}^0)$ is equivalent in $(\mathcal{U}, P, a^0)$ to an $L_{\omega_1\omega}$-formula $\varphi(\bar{x})$ without parameters.

To obtain $\varphi$ in (C), convert in $\theta$ each $y$-quantifier, $\forall y \cdots y \cdots$ into $\bigwedge_{a \in A} \cdots a \cdots$, with $a$ a new constant for $a \in A$, and similarly for $\exists y$. Then replace each resulting atomic formula $\pi(\bar{a})$, $P\bar{a}$ by its actual truth-value in $(\mathcal{U}, P)$.

**Theorem (D)** (Motohashi [Mo]). For any $\sigma$ in $L_{\omega_1\omega}(P)$, (i) \iff (ii).

(i) For all (or, for all countable) $(\mathcal{U}, P) \models \sigma$, $|M(\mathcal{U}, P)| = 1$.

(ii) $\sigma \models \forall \bar{x}[P\bar{x} \leftrightarrow \theta(\bar{x})]$ for some $M$-formula $\theta(\bar{x})$.

By (C), (ii) obviously implies (i). (D) can be proved by an application of Feferman’s many-sorted interpolation theorem [F]. This proof as well as Motohashi’s proof in [Mo] gives the result for all admissible fragments of $L_{\omega_1\omega}$. Hence by (B), (D) implies Svenonius’ theorem [Sv]. Also by (C), (D) implies Scott’s definability theorem [Sc] (apply (D) for the Scott sentence $\sigma$ of $(\mathcal{U}, P)$).

**Theorem 2.** For any sentence $\sigma$ in $L_{\omega_1\omega}(P)$, (i) \iff (ii).

(i) For all countable $(\mathcal{U}, P) \models \sigma$, we have $|M(\mathcal{U}, P)| \leq \aleph_0$ (or, $< 2^{\aleph_0}$).

(ii) $\sigma \models \forall \bar{x} \exists \bar{y} \forall \bar{x}[P\bar{x} \leftrightarrow \theta_{i}(\bar{x}, \bar{x}^i, \bar{y}^i)]$ for some $M$-formulas $\theta_{i}(i < \omega)$.

By (C), (ii) obviously implies (i). For the same reason, Theorem 2 specializes to (A) if $\sigma$ is the Scott sentence of $(\mathcal{U}, P)$. By (B), Theorem 2 for finitary logic is a form of the weak-definability theorem [C], [Ma 1], [R]. As Motohashi [Mo] shows, conditions (i) in Theorems 1 and 2 are
not equivalent for $L_{\omega_1\omega}(P)$, unlike in the finitary case. In fact, even (i) in (D) does not imply (i) in Theorem 1.

**Theorem 3.** For any sentence $\sigma$ in $L_{\omega_1\omega}$, (i) $\iff$ (ii).

(i) For all countable $\mathfrak{U} \models \sigma$, $\mathfrak{U}$ has at most countably many (or equivalently, less than $2^{\aleph_0}$) automorphisms.

(ii) $\sigma \models \forall_{i<\omega} \exists x^i \forall y \forall x[x = y \leftrightarrow \theta_i(x, y, x^i, y^i)]$ for some $M$-formulas $\theta_i (i < \omega)$ without $P$.

By Proposition (C), in any given $\mathfrak{U}$, the part after "\forall y" of the formula in (ii) implies that $y$ is definable in $\mathfrak{U}$ with the parameters $x^i$. Hence Theorem 3 has the following

**Corollary (E) (Kueker [Ku 1]).** For any countable $\mathfrak{U}$, $\mathfrak{U}$ has at most countably many (less than $2^{\aleph_0}$) automorphisms iff there are some finitely many elements $a$ of $A$ such that every element of $A$ is definable in $(\mathfrak{U}, a)$ by an $L_{\omega_1\omega}$-formula.

The finitary version of Theorem 3 is, via (B), the well-known result that every finitary sentence with infinite models has a countable model with $2^{\aleph_0}$ automorphisms.

Our last result utilizes, and adds to, Kueker’s work on “finite generalizations” of Beth’s theorem [Ku 2].

**Theorem 4.** For any $\sigma$ in $L_{\omega_1\omega}(P)$, (i) $\iff$ (ii).

(i) For all (or, for all countable) $\mathfrak{U}$, $|M_{\sigma}(\mathfrak{U})| < \aleph_0$.

(ii) $\sigma \models \forall_{n<\omega} \exists \bar{\phi}_n(\bar{v}^n) \land \forall \bar{v}^n [\phi_n(\bar{v}^n) \rightarrow \forall_{i<n} \forall x[Px \leftrightarrow \phi_n,i(x, \bar{v}^n)]]$ for some $\phi_n,i(x, \bar{v}^n)$ in $L_{\omega_1\omega}$.

2. **Proofs.** The proofs use abstract consistency properties (see [Ke], [Ma 2], [Ma 3]) and in case of Theorems 2 and 3, approximation of automorphisms by finite pieces similarly as in the proofs in [Ma 3]. We will show the proof of Theorem 2 in some detail.

**Proof of Theorem 2.** Let $C$ be a countably infinite set of new individual constants. Define $\Gamma_2$ to be the collection of objects $\gamma = \langle s, f_i \rangle_{i \in I}$ such that $s$ is a finite set of sentences of $L_{\omega_1\omega}(P)(C)$ in negation normal form (n.n.f.) with only finitely many constants from $C$, $I$ is a finite set, each $f_i$ is a finite subset of $C \times C$, and such that (the main condition) there is no formula $\mu$ with (i)$_2(\gamma, \mu)$ where:

(i)$_2(\gamma, \mu)$ is of the form of the formula after “$\models$” in Theorem 2(ii) and whenever $(\mathfrak{U}, P, \bar{c}) \in C$ is a model of $s$ and, for $i \in I$, $g_i$ is an automorphism of $\mathfrak{U}$ such that $\langle c, d \rangle \in f_i \Rightarrow \langle \bar{c}, \bar{d} \rangle \in g_i$, then $\mathfrak{U} \models \mu$.

Suppose $\sigma$ is in n.n.f. and it does not satisfy (ii) in Theorem 2. Then clearly $\gamma_0 = \sigma \langle \{ \sigma \} \rangle \notin \mathfrak{U}$ belongs to $\Gamma_2$. We successively extend this element
of $\Gamma_2$, always remaining in $\Gamma_2$, such that the limit of the procedure yields, in a natural way, a model $(\mathfrak{M}, P)$ with $|M(\mathfrak{M}, P)| = 2^{\omega_0}$.

**Lemma (ii).** For fixed $I$ and $f_i$ ($i \in I$), \{s; $\langle s, f_i \rangle_{i \in I} \in \Gamma_2$\} is an abstract consistency property.

(iii) For any $\gamma = \langle s, f_i \rangle_{i \in I} \in \Gamma_2$, $j \in I$, $c \in C$, let $d \neq c$ and let $d$ not occur in $\gamma$. Then

$$\langle s, f_i, f_j \cup \{\langle c, d \rangle \} \rangle_{i \in I \setminus \{j\}} \quad \text{and} \quad \langle s, f_i, f_j \cup \{\langle d, c \rangle \} \rangle_{i \in I \setminus \{j\}}$$

belong to $\Gamma_2$.

*Comment.* (iii) will be used to make sure that the domains and ranges of purported automorphisms will indeed be the whole domain (in this case, essentially $C$) of the structure.

(iv) Let $\gamma$ and $i$ be as in (iii). Let $c$, $d_1$, $d_2$ be distinct constants in $C$ but not in $\gamma$. Put $f_j' = f_j \cup \{\langle d_1, c \rangle \}$, $f_j'' = f_j \cup \{\langle d_2, c \rangle \}$ and $s' = s \cup \{Pd_1, \neg Pd_2\}$. Then $\gamma' = \langle s', f_i, f_j', f_j'' \rangle_{i \in I \setminus \{j\}} \in \Gamma_2$.

*Comment.* (iv) is used to “split” a finite approximation $f_j$ into two. Eventually the infinite paths of the tree of such approximations will be the automorphisms and they will give us $2^{\omega_0}$ images of $P$. Note that for “extensions” $g_1', g_2'$ of $f_1', f_2'$, resp., "$g_1P \neq g_2''P$".

**Proof of (iv).** Introduce new operation symbols $g_i$ ($i \in I$). The assumption that $\gamma' \notin \Gamma_2$ leads to the existence of $\mu'$ with $(ii_2(\gamma', \mu'))$. Let $\xi$ be the formula $\neg \mu' \land \bigwedge_i s \land \bigwedge_i g_i \text{ "is an L-automorphism extending } f_i'\text{"}$. By $(i_2)(\gamma', \mu')$

(v) $(\mathfrak{M}, P, \tilde{c}, g_i)_{c \in \text{dom } f_j} \models \xi$ implies that every automorphism of $(\mathfrak{M}, \tilde{c})_{c \in \text{dom } f_j}$ leaves $P$ fixed.

Hence by (an inessential strengthening of) (D),

(vi) $\xi \models \forall x[P\tilde{x} \leftrightarrow \theta(\tilde{x}, \tilde{c})]$ for $\tilde{c} = \text{dom } f_j$ and for some Motohashi formula $\theta(\tilde{x}, \tilde{x}', \tilde{y}')$. Hence $\xi \models \mu''$ where $\mu'' = \exists \tilde{x}' \tilde{y}' \forall \tilde{x} [P\tilde{x} \leftrightarrow \theta(\tilde{x}, \tilde{x}', \tilde{y}')]$. It follows that

(i) $2(\gamma', \mu' \lor \mu')$ holds, contrary to $\gamma \in \Gamma_2$.

Now, let $I_n$ be the set of finite 0-1 sequences of length $n$. Let $C = \{c_n: n < \omega\}$. We construct a sequence $\gamma_n$ ($n < \omega$) of elements of $\Gamma_2$ starting with $\gamma_0 = \langle \{\sigma\}, \emptyset \rangle$ such that $\gamma_n = \langle s_n, f^n_i \rangle_{i \in I_n}$, $s_n \subseteq s_{n+1}$, $f^n_i \subseteq f^{n+1}_i$ for $j = i \cap \langle 0 \rangle, i \cap \langle 1 \rangle$ and

(vii) $s_0 = \bigcup_{n < \omega} s_n$ is pseudocomplete (see 1.3 Definition in [Ma 2]) or, what is the same, the $s_n$ satisfy (1)-(5) on p. 13 in [Ke] (here we use (iii)),

(viii) $c_n \in \text{dom } f^n_{i_0+1} \cap \text{rn } f^n_{i_1+1} (i \in I_n)$ (here we use (iii)), and

(ix) for each $n$ and $i \in I_n$, there are $d_0 \in \text{dom } f^n_{j_0+1}, d_1 \in \text{dom } f^n_{j_1+1}$ and $c \in \text{rn } f^n_{j_0+1} \cap \text{rn } f^n_{j_1+1}$ (here $j_0 = i \cap \langle 0 \rangle$, $j_1 = i \cap \langle 1 \rangle$) such that $\{Pd_0, \neg Pd_1\} \subseteq s_{n+1}$ (here we use (iv)).
For the canonical model $\langle \mathfrak{U}, P, \mathcal{C} \rangle$ of $s_\alpha$ (see the proof of the model existence theorem in [Ke], or 1.4 in [Ma 3]) we have

$(x)$ $\mathfrak{U} \models \sigma$,

$(xi)$ the maps $f_a = \{ (c, d) : (c, d) \in \bigcup_{n<\omega} f_a^n \}$ for $\alpha = \omega$ are automorphisms of $\mathfrak{U}$ (mainly by (viii)) and

$(xii)$ $f_a P \neq f_a P$ for $\alpha \neq \alpha'$ by (ix). Q.E.D.

**ON THE PROOF OF THEOREM 1.** The collection playing the role of $\Gamma_2$ above, $\Gamma_1$, is defined as follows. Let $P$ denote distinct predicate symbols of the same arity as $P$, and let us write $s(P_i)$ for a set of sentences in $L_{\omega_1 \omega}(P_i(C))$. Let $\Delta_I$ be the set of sentences of the form

$$\bigvee_{i \in I} \forall \forall \exists \bar{x} \ [P_i \bar{x} \leftrightarrow \phi_n^i(\bar{x}, \bar{u})]$$

where the $\phi_n^i$ are in $L_{\omega_1 \omega}$. We define $\Gamma_1$ to be the collection of objects $\gamma = \langle s(P_i) \rangle_{i \in I}$ with similar finiteness conditions as for $\Gamma_2$ and such that there is no $\mu$ with (i)$(\gamma, \mu)$ where:

(i)$(\gamma, \mu)$ $\mu \in \Delta_I$ and $\bigcup_{i \in I} s(P_i) = \mu$.

The crucial fact analogous to (iv) above is that for $\gamma$ as above, and a fixed $j \in I$, if we put $s_j'(P_j) = s_j(P_j) \cup \{ P_j c, \neg (s_j(P_j) \cup \{ \neg P_j c \}$ with $c \in C$ a constant not in $\gamma$, then $\langle s_j(P_j), s_j'(P_j) \rangle_{i \in I \ldots (j)}$ again belongs to $\Gamma_1$. The proof of this applies the Beth-Lopez-Escobar theorem.

**ON THE PROOF OF THEOREM 3.** It is very similar to that of Theorem 2 and applies a corollary to (D); if every model of $\sigma$ has no nontrivial automorphisms, then $\sigma \models \forall y \forall x [x = y \leftrightarrow \theta(x, y)]$ for some $M$-formula $\theta$ without $P$.

**ON THE PROOF OF THEOREM 4.** Let us call a formula of the form after "$\models$" in Theorem 4 (ii) a $K$-formula. Consider $\sigma = \sigma(P)$ not satisfying (ii). Define $\Gamma_4 = S_4$ to be the set of sets $s(P_0, \ldots, P_{n-1})$ of sentences of $L_{\omega_1 \omega}(P_0, \ldots, P_{n-1}) (C)$ with the usual finiteness conditions such that for any $K$-formula $\kappa(P)$, $s \not\models \sigma(P) \rightarrow \kappa(P)$. The crucial property of $S_4$ is that if $s \in S_4$ is as above then $s \cup \{ s(P_n), \ "P_n \neq P_1", \ "P_n \neq P_2", \ldots, \ "P_n \neq P_{n-1}" \}$ belongs to $S_4$. Also, $S_4$ is an abstract consistency property.

**ADDED IN PROOF (May 2, 1973).** Jon Barwise noticed that Theorem 1 remains true if we replace $\sigma$ by a $\Sigma_1^1$-over-$L_{\omega_1 \omega}(P)$ sentence $\exists \bar{S} \sigma(P, \bar{S})$. A similar remark holds for the rest of the theorems too. In fact, no essential change is required in the proofs. Barwise also noticed that from the $\Sigma_1^1$ generalization of Theorem 1 in the "admissible version," the following strengthening of a theorem due to J. Harrison results immediately: If a $\Sigma_1^1$ set of reals does not contain a perfect subset, it is a subset of a set constructible below $\omega_1^K$ (Kleene's $\omega_1$) (notice that our proof gives in fact a perfect subset of $M_{\omega_1}(\mathfrak{U})$). Subsequently, the author noticed that the $\Sigma_1^1$ generalization of Theorem 1 (formulated with "perfect subset") combined...
with an approximation theorem of Vaught (any constructible $\Pi^1_1$-over-$L_{\omega_1,\omega}$ sentence is equivalent for countable structures to $\bigvee_{\delta \in \omega_1} \delta \delta$ with some constructible sequence $\langle \delta : \alpha < \omega_1 \rangle$ of $L_{\omega_1,\omega}$-sentences) directly (and without the use of forcing) gives Mansfield's theorem: any $\Sigma^1_2$ set of reals not containing a perfect subset is constructible.

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