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AN INTEGRAL RELATED TO NUMERICAL INTEGRATION

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For real- or complex-valued functions defined on a finite real interval, the concept of integral that is most suitable for numerical approximation is that of Riemann. There the integral is defined as a limit of Riemann sums, any of which can be effectively calculated (given the calculability of the integrand). In fact most common quadrature rules—such as the trapezoid rule, Simpson's rule, or the Gauss-Legendre formulas—do in fact approximate the integral by calculating carefully chosen Riemann sums. Each of these rules converges for the full class of (properly) Riemann integrable functions; there seems to be no larger interesting class of bounded functions for which any quadrature rules converge.

For infinite intervals the situation is not so neat. The improper Riemann integral over $[0, \infty)$ is not defined as a limit of finite sums, and indeed there is no sequence of quadrature formulas

$$Q_n(f) = \sum_{r=1}^{n} a_{r,n} f(x_{r,n})$$

having the property that $Q_n(f) \to \int_0^\infty f$ whenever f is improperly Riemann integrable.[1]

What can we hope for? If we wish to exhibit $\int_0^\infty f$ as a limit of Riemann sums, clearly those sums must be based on partitions of intervals that expand to fill $[0, \infty)$. Furthermore the gauges of those partitions—the lengths of their longest subintervals—must simultaneously go to zero; otherwise we would not get the correct integral even for functions that are zero outside a finite interval.

DEFINITION. A complex-valued function f, defined on $[0, \infty)$, will be called "simply integrable" if there is a number I with the following property: For every $\varepsilon > 0$ there are numbers $B = B(\varepsilon)$ and $\Delta = \Delta(\varepsilon)$ such that if $b > B$ and $\Pi: 0 = x_0 < x_1 < \cdots < x_n = b$ is any partition of $[0, b]$ with $\max\{(x_r - x_{r-1})\} < \Delta$ and $\xi_1, \xi_2, \ldots, \xi_n$ are any numbers satisfying

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[1] In this paper the improper Riemann integral $\int_0^\infty f(x)dx$ is understood as the finite limit of the proper Riemann integral $\int_0^\infty f(x)dx$ as $b \to \infty$.

$\xi_r \in [x_{r-1}, x_r]$, $r = 1, 2, \ldots, n$, then

$$\left| \sum_{r=1}^{n} (x_r - x_{r-1}) f(\xi_r) - I \right| < \varepsilon.$$

The number I (which is obviously unique) is the "simple integral" of f.

Clearly, if f is simply integrable then it is improperly Riemann integrable and $I = \int_0^\infty f$. (The interval $(-\infty, \infty)$, and any other half-infinite interval, can be handled similarly; we will deal only with $[0, \infty)$.)

The relation of the simple integral to quadrature formulas is given in Theorem 3. We first characterize simple integrability, in Theorems 1 and 2. Proofs of these theorems are to appear in [**3**], among other results.

DEFINITION. If δ is a positive number, an increasing sequence $S = \{x_0, x_1, \ldots, x_n\}$ of nonnegative real numbers is called "δ-separated" if $x_r - x_{r-1} \geqq \delta$ for every r. If f is defined on S, the quantity $\sum_{r=1}^{n} |f(x_r) - f(x_{r-1})|$ is called "the variation of f on S".

If f is defined on $[0, \infty)$, then the "δ-variation of f" ("$V_\delta(f)$") is the supremum of the variations of f on all δ-separated sequences. If $V_\delta(f)$ is finite for every $\delta > 0$, then f is said to be "of bounded coarse variation" ("BCV").

THEOREM 1. *If f is improperly integrable over $[0, \infty)$ then it is simply integrable if and only if it is of BCV.*

A function that is of BV on $[0, \infty)$ is of BCV; but BCV, unlike BV, places no restriction (other than boundedness) on the values of the function in any finite interval.

So BV is a sufficient condition for simple integrability (for functions that are improperly integrable). Other sufficient conditions may be found: for example, if f is monotonic on $[0, \infty)$ and is improperly integrable, and g is improperly integrable and $|g(x)| \leqq |f(x)|$ for all x, then g is simply integrable. $(\sin x)/x$ is not simply integrable; $(\sin x^2)/x^2$ is, though it is not of BV.

DEFINITION. Let ε be any positive number. A real-valued function f, defined on an interval I, is "ε-increasing" on I if $f(y) \geqq f(x)$ whenever x and y are points of I with $y \geqq x + \varepsilon$.

THEOREM 2. *Let f be a real-valued function on $[0, \infty)$ that is bounded on every finite interval. Then for every $\varepsilon > 0$, f is a difference of two functions that are ε-increasing on $[0, \infty)$. f is of BCV if and only if it is, for every $\varepsilon > 0$, a difference of two functions that are ε-increasing and bounded on $[0, \infty)$.*

A quadrature formula

$$Q(f) = \sum_{r=1}^{n} a_r f(x_r) \approx \int_a^b f(x)\, dx$$

defines a Riemann sum for the integral it is approximating when:

(1) $a_r > 0, r = 1, 2, \ldots, n$,

(2) $a + a_1 + \cdots + a_{r-1} \leqq x_r \leqq a + a_1 + \cdots + a_r, r = 1, 2, \cdots, n$ and

(3) $a_1 + a_2 + \cdots + a_n = b - a$.

The partition involved is $a = t_0 < t_1 < \cdots < t_n = b$, where $t_1 = a + a_1$, $t_2 = a + a_1 + a_2$, etc. The gauge of the partition is the largest of the numbers a_r. If the same formula is applied to another interval $[c, d]$, via an affine change of variable, the coefficients a_r are multiplied by $(d - c)/(b - a)$; and so is the gauge.

Now say $Q_1, Q_2, \ldots$ is a sequence of quadrature formulas that define Riemann sums, with largest coefficients $A_1, A_2, \ldots$ respectively when applied on the interval $[0, 1]$ or $[-1, 1]$. For example, if Q_n is the $(n + 1)$-point trapezoid formula applied to $[0, 1]$, $A_n = 1/n$; if it is the n-point Gauss-Legendre formula applied to $[-1, 1]$, A_n is asymptotic to π/n. (That the Gauss-Legendre formulas define Riemann sums was shown by Stieltjes [**1**]; for the asymptotic estimate of A_n see, e.g. [**2**, p. 350].) Let $Q_n(f, b_n)$ denote the result of applying Q_n to the integration of f over $[0, b_n]$.

THEOREM 3. *With Q_n and A_n as above, if f is simply integrable then $\lim_{n\to\infty} Q_n(f, b_n) = \int_0^\infty f$ as long as*

(1) $\lim_{n\to\infty} b_n = \infty$, and

(2) $\lim_{n\to\infty} b_n A_n = 0$.

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