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ESTIMATES FOR WEAK-TYPE OPERATORS

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1. **Introduction.** When f is an integrable function on the interval $[0, 1]$, we denote by f^* its nonincreasing rearrangement and by f^{**} the average $f^{**}(t) = (1/t)\int_0^t f^*(s)\,ds$. The Lorentz space L^{pq}, $1 \leqq p \leqq \infty$, $1 \leqq q \leqq \infty$ consists of all functions f for which the norm

$$\|f\| = \left\{\int_0^1 [t^{1/p}f^{**}(t)]^q \frac{dt}{t}\right\}^{1/q}$$

is finite; the Lorentz space $L^{(pq)}$ is defined in the same way except that f^{**} is replaced by f^*. When $1 < p \leqq \infty$, L^{pq} and $L^{(pq)}$ coincide, up to equivalence of (quasi) norms (cf. [3], [4]). The spaces L^{pq}, $1 < p < \infty$, are the intermediate spaces $(L^1, L^\infty)_{\theta,q;K}$, $\theta = 1 - 1/p$, generated by the K-method of J. Peetre (cf. [2], [5], [7]). Note that $L^{(1\infty)}$ is the space usually referred to as "weak-L^1" and that the Orlicz space $L \log^+ L$ of functions f for which $|f| \log^+ |f|$ is integrable, is (cf. [1]) none other than the Lorentz space L^{11}. Thus

$$L \log^+ L = L^{11} \subseteq L^{1\infty} = L^1 = L^{(11)} \subseteq L^{(1\infty)} = \text{weak-}L^1.$$

From characterizations of the intermediate spaces $(L \log^+ L, L^1)_{\theta,q;K}$ and $(L^1, \text{weak-}L^1)_{\theta,q;K}$ obtained by the author in [1] and subsequently, there follow some new estimates for weak-type operators. In particular, we obtain a sharper form of a theorem of O'Neil [6] concerning operators that are simultaneously of weak-types (1, 1) and (p, p), $1 < p \leqq \infty$.

2. **Intermediate spaces between $L \log^+ L$ and L^1.** The space of functions f for which the norm

$$\|f\| = \left\{\int_0^1 \left[t\left(\log \frac{1}{t}\right)^{\theta - 1/q} f^{**}(t)\right]^q \frac{dt}{t}\right\}^{1/q}$$

is finite, will be denoted by $A^{\theta q}$, $0 < \theta < 1$, $1 \leqq q \leqq \infty$. The corresponding space with f^{**} replaced by f^* in the previous definition, is denoted by $A^{(\theta q)}$. The following results were obtained by the author in [1]:

THEOREM 1. $(L^1, L \log^+ L)_{\theta,q;K} = A^{\theta q}$, $0 < \theta < 1$, $1 \leqq q \leqq \infty$.

COROLLARY 1.1. $(L^1, L \log^+ L)_{\theta,1;K} = L(\log^+ L)^\theta$, $0 < \theta < 1$.

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COROLLARY 1.2. $(L^1, L\log^+ L)_{1/q,q;K} = L^{1q}$, $1 < q < \infty$.

Also characterized in [1] are the spaces $(L\log^+ L, L^\infty)_{\theta,q;K}$. We have no need here of any explicit characterization but let us note the following result.

THEOREM 2. $(L\log^+ L, L^\infty)_{\theta,q;K} \subsetneqq L^{pq}$, $\theta = 1 - 1/p$.

The proofs of all these results depend crucially on the fact that the Orlicz space $L\log^+ L$ is also a Lorentz Λ-space (cf. [1]). In the next section when we consider the space weak-L^1, the situation is radically different and the same techniques do not apply.

3. **Intermediate spaces between L^1 and weak-L^1.** For a measurable function f on $[0, 1]$, $f^\#$ denotes the nonincreasing rearrangement on $(0, \infty)$ of the function $tf^*(t)$, taken with respect to the measure $dm^*(t) = dt/t$. Thus $f^\#$ is the right-continuous inverse of the distribution function $\sigma \to m^*\{t : tf^*(t) > \sigma\}$. The next theorem shows that the intermediate spaces between L^1 and weak-L^1 are simply the "Lorentz spaces with respect to $f^\#$".

THEOREM 3. *A necessary and sufficient condition that f belong to $(L^1, \text{weak-}L^1)_{\theta,q;K}$, $0 < \theta < 1$, $1 \leqq q \leqq \infty$, is that the quasinorm*

$$\|f\| = \left\{\int_0^\infty [t^{1-\theta} f^\#(t)]^q \frac{dt}{t}\right\}^{1/q}$$

be finite.

It is not difficult to check that the quasinorm in the statement of Theorem 3 dominates the quasinorm of the space $A^{(1-\theta,q)}$. Thus

COROLLARY 3.1. $(L^1, \text{weak-}L^1)_{\theta,q;K} \subseteq A^{(1-\theta,q)}$.

The Peetre K-functional norm $K(t; f) = K(t; f; \text{weak-}L^1, L^\infty)$ (cf. [2], [5], [8]) for the pair (weak-L^1, L^∞) is given by $K(t; f) = \sup_{0<s<t} sf^*(s)$. Since $f^*(t) \leqq t^{-1}K(t; f) \leqq f^{**}(t)$, it follows that

$$L^{pq} \subseteq (\text{weak-}L^1, L^\infty)_{\theta,q;K} \subseteq L^{(pq)},$$

$\theta = 1 - 1/p$. Hence

THEOREM 4. $(\text{weak-}L^1, L^\infty)_{\theta,q;K} = L^{pq}$, $\theta = 1 - 1/p$, $1 < p < \infty$, $1 \leqq q \leqq \infty$.

4. **The interpolation theorems.** We are now in a position to exhibit the interpolation theorems corresponding to the various classes of intermediate spaces described in §§2 and 3. Our main result is:

THEOREM 5. *Let T be a quasilinear (cf. [5]) operator, $T: L\log^+ L \to L^1$ and $T: L^1 \to$ weak-L^1. Then $T: A^{\theta q} \to A^{(\theta q)}$, $0 < \theta < 1$, $1 \leqq q \leqq \infty$.*

The cases $\theta = 1/q$ and $q = 1$ yield the following corollaries:

COROLLARY 5.1. *Under the hypotheses of Theorem 5, T is a bounded operator from the Lorentz space L^{1q} into the Lorentz space $L^{(1q)}$, $1 \leqq q \leqq \infty$.*

COROLLARY 5.2. *Under the hypotheses of Theorem 5, T is a bounded operator from the Orlicz space $L(\log^+ L)^\theta$ into the space $A^{(\theta 1)}$, $0 < \theta < 1$.*

Any operator T of weak-types $(1, 1)$ and (p, p), $p > 1$, will satisfy the hypotheses of Theorem 5 (cf. [6]) and hence the conclusions above; in particular, any such operator maps $L(\log^+ L)^\theta$ into $A^{(\theta 1)}$. Since $A^{(\theta 1)} \subsetneqq A^{(\theta, 1/\theta)} = L^{(1, 1/\theta)}$, Corollary 5.2 is sharper than the following result of O'Neil [6]:

COROLLARY 5.3 (O'NEIL). *If T is of weak-types $(1, 1)$ and (p, p), $p > 1$, then $T: L(\log^+ L)^\theta \to L^{(1, 1/\theta)}$.*

Finally, let us note that if we combine Theorem 4 and the fact that the L^{pq} spaces are the intermediate spaces between L^1 and L^∞, we can reproduce the following special case of the Marcinkiewicz-Calderón-Hunt theorem (cf. [3], [4], [5], [8]).

THEOREM 6 (MARCINKIEWICZ-CALDERÓN-HUNT). *If T is of weak-types $(1, 1)$ and (∞, ∞), then $T: L^{pq} \to L^{pq}$, $1 < p < \infty$, $1 \leqq q \leqq \infty$.*

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