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EXTREMAL POSITIVE SPLINES WITH APPLICATIONS TO INTERPOLATION AND APPROXIMATION BY GENERALIZED CONVEX FUNCTIONS

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1. **Preliminaries.** Consider an nth order Chebyshevian, or disconjugate, differential operator $L = w_0^{-1}Dw_1^{-1} \cdots Dw_n^{-1}$, where $D = d/dt$ and $w_i \in C^n(\boldsymbol{R})$, $w_i(t) > 0$, $i = 0, 1, \ldots, n$, $t \in \boldsymbol{R}$ [4], [5]. The formal adjoint of L is the Chebyshevian operator $L^* = (-1)^n w_n^{-1}Dw_{n-1}^{-1} \cdots Dw_0^{-1}$. Let

$$F(s, t) = w_n(s)w_0(t)\int_t^s dt_1 w_1(t_1)\int_{t_1}^s dt_2 w_2(t_2) \cdots \int_{t_{n-2}}^s dt_{n-1}w_{n-1}(t_{n-1}).$$

A fundamental solution for L is given by $G(s, t) = F(s, t)$ for $s \geqq t$, $G(s, t) = 0$ for $s < t$. A fundamental solution for L^* is $G_*(s, t) = G(t, s)$. To avoid cumbersome formulations results are stated for $n \geqq 2$ (in which case G is continuous), unless indicated otherwise.

By $\mathfrak{M}(T)$ we mean the collection of Radon measures on the locally compact Hausdorff space T; by $\mathfrak{M}_0(T)$ and $\mathfrak{M}(T)^+$, the subfamilies of measures of compact support and of positive measures. For an open interval I let $\mathfrak{M}^n(I)$ be the set of real functions on I possessing an nth distribution derivative belonging to $\mathfrak{M}(I)$, $n = 1, 2, \ldots$. For $n \geqq 2$, if $u \in \mathfrak{M}^n(I)$ then $D^{n-2}u \in AC^{\mathrm{loc}}(I)$ and $D^{n-1}u \in BV^{\mathrm{loc}}(I)$. One shows that a measure $u \in \mathfrak{M}(I)$ belongs to $\mathfrak{M}^n(I)$ iff $Lu \in \mathfrak{M}(I)$ in the following weak sense: There is $\mu \in \mathfrak{M}(I)$ such that $\int L^*\phi(t)u(dt) = \int \phi(t)\mu(dt)$ for each $\phi \in C_0^n(I)$; if this is so one says $\mu = Lu$. Let $\mathfrak{M}_0^n(\boldsymbol{R})$ consist of the functions in $\mathfrak{M}^n(\boldsymbol{R})$ of compact support. For any interval I we say $u \in \mathfrak{M}_{00}^n(I)$ if $u \in \mathfrak{M}_0^n(\boldsymbol{R})$ and $\operatorname{supp}(u) \subset I$. Each $u \in \mathfrak{M}^n(I)$ has integral representations

$$u(s) = v(s) + \int_a^s F(s, t)Lu(dt), \quad \text{where } Lv = 0 \text{ and } a \in I,$$

and analogously

$$u(t) = v^*(t) + \int_a^t F(s, t)L^*u(ds), \quad \text{with } L^*v^* = 0.$$

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Let $\ker^* L = \{\lambda \in \mathfrak{M}_0(\boldsymbol{R}) : \int u\lambda = 0 \text{ if } Lu = 0\}$. It is fairly standard in distribution theory that the following three properties are mutually equivalent:

(1) $\lambda \in \ker^* L$;

(2) $\phi(t) = \int G(s, t)\lambda(ds)$ has compact support;

(3) there is $\phi \in \mathfrak{M}_0(\boldsymbol{R})$ such that $L^*\phi = \lambda$.

Also, L^* maps $\mathfrak{M}_0^n(\boldsymbol{R})$ one-to-one onto $\ker^* L$. The inverse map is given by $\lambda \mapsto \phi$, $\phi(t) = \int G(s, t)\lambda(ds)$.

Now let I be any interval and $\Omega = \operatorname{int}(I)$. We define the Chebyshevian convexity cone $\mathscr{K}(L, I) = \{u \in C(I) : L(u \mid \Omega) \in \mathfrak{M}^+(\Omega)\}$. The dual cone is $\mathscr{K}^*(L, I) = \{\lambda \in \mathfrak{M}_0(I) : \int u\lambda \geqq 0 \text{ for all } u \in \mathscr{K}(L, I)\}$ [5], [9], [10]. The known characterization [4], [10] of the dual cone combined with the above characterization of $\ker^* L$ almost gives Lemma 1:

LEMMA 1. (a) *Let* $\lambda \in \mathfrak{M}_0(I)$. *Then* $\lambda \in \mathscr{K}^*(L, I)$ *iff there is* $\phi \in \mathfrak{M}_{00}^n(I)$, $\phi \geqq 0$, $\lambda = L^*\phi$; *moreover, for such* λ *and* ϕ *and for* $u \in \mathscr{K}(L, I)$, *one has* $\int u\lambda = \int \phi Lu$.

(b) L^* *maps* $\mathfrak{M}_{00}^n(I)^+$ *one-to-one onto* $\mathscr{K}^*(L, I)$. (*Here, as throughout, the superscript* $^+$ *denotes the positive cone in a space of functions.*)

The proof for the case that I is not open depends on a subtle application of Fubini's theorem and the total positivity properties of the fundamental solution $G(s, t)$ [5], [7], [1].

2. **P-extremal measures in** $\mathscr{K}^*(L, I)$.

DEFINITION. Let $\mathscr{C}$ be a convex cone in a linear space and let $0 \neq \lambda \in \mathscr{C}$. We say λ is *decomposable*, if there is a *decomposition* $\lambda = \lambda_1 + \lambda_2$ with linearly independent $\lambda_1, \lambda_2 \in \mathscr{C}$. Otherwise λ is *extremal in* $\mathscr{C}$. If $\mathscr{C} \subset \mathfrak{M}_0(T)$, T locally compact Hausdorff, then λ is *P-decomposable* (*ν-decomposable* for some $\nu \in \mathfrak{M}(T)$) in $\mathscr{C}$ if there is a decomposition $\lambda = \lambda_1 + \lambda_2$ in $\mathscr{C}$ such that $\int |\lambda| = \int |\lambda_1| + \int |\lambda_2|$ ($\lambda_i \ll \nu$); otherwise λ is *P-extremal* (*ν-extremal*) in $\mathscr{C}$.

REMARK. All nonzero λ in $\mathscr{K}^*(L, I)$ are decomposable.

LEMMA 2. *Let* $0 \neq \lambda \in \mathscr{C} \subset \mathfrak{M}_0(T)$. *Then* $\lambda = \lambda_1 + \lambda_2$ *is a P-decomposition in* $\mathscr{C}$ *iff there are two nonconstant functions* $h_i \in L^1(|\lambda|)$, $0 \leqq h_i$, $h_1 + h_2 = 1$ *a.e.* $|\lambda|$ *and* $\lambda_i = h_i\lambda \in \mathscr{C}$ $(i = 1, 2)$. *In particular, if* λ *is* $|\lambda|$-*extremal in* $\mathscr{C}$ *then* λ *is P-extremal in* $\mathscr{C}$.

DEFINITIONS. An r-fold zero of $f \in \mathfrak{M}^n(I)$ is defined as usual for $0 \leqq r \leqq n - 2$. Any maximal interval on which f vanishes will be termed a zero-interval and is counted as n-fold zero. If $f(t) = f'(t) = \cdots = f^{(n-2)}(t) = 0$, let $A = f^{(n-1)}(t-)$, $B = f^{(n-1)}(t+)$. If $AB > 0$ then t is an $(n - 1)$-fold zero of f; if $AB \leqq 0$ the multiplicity is n. The total number of zeroes of f is denoted by $Z^n(f, I)$. Let $s_1 < s_2 < \cdots < s_p$. We denote by $S(L, s_1, \ldots, s_p)$ the space of *Chebyshevian L-splines* with knots $s_1, \ldots, s_p$, i.e. of functions $u \in \mathfrak{M}^n(\boldsymbol{R})$ for which Lu has support contained in $\{s_1, \ldots, s_p\}$. The space of Chebyshevian L^*-splines of compact support with knots $s_1, \ldots, s_p$ is denoted by $S_0(L^*, s_1, \ldots, s_p)$. It consists of functions ϕ of the form $\phi(t) = \sum_{i=1}^p \alpha_i G(s_i, t)$ where $\sum_{i=1}^p \alpha_i u(s_i) = 0$ whenever $u \in \ker L$. Finally let $S^-(\lambda)$ denote the number of sign changes of the Radon measure λ in $\mathfrak{M}_0(I)$. Cf. **[5]**, **[6]**, **[7]**, **[8]** for results relating to these definitions.

We can now state the central result of this paper.

THEOREM 1. (i) *Let $s_1 < s_2 < \cdots < s_p$ and $t_1 \leqq t_2 \leqq \cdots \leqq t_{p-n-1}$ be any reals selected subject to $p \geqq n + 1$ and*

(4) $s_{i+1} < t_i < s_{i+n}, i = 1, \ldots, p - n - 1$;

(5) *each distinct t_i occurs with even multiplicity not larger than $n - 1$.*

Then there is a Chebyshevian L^*-spline ϕ, unique up to a positive factor, satisfying

(6) $\phi \in S_0(L^*, s_1, \ldots, s_p)^+, \phi \neq 0$.

(7) $Z^n(\phi, (s_1, s_p)) = p - n - 1$ *and ϕ vanishes at each distinct t_i according to the multiplicity of its occurrence.*

Moreover, one such function can be explicitly represented by the (extended) determinant

$$(8) \quad \phi(t) = G^*\begin{pmatrix} s_1, & \ldots, & s_n, & s_{n+1}, & \ldots, & s_{p-1}, & s_p \\ r_1, & \ldots, & r_n, & t_1, & \ldots, & t_{p-n-1}, & t \end{pmatrix} \qquad (t \neq t_i).$$

The r_i are any reals satisfying $r_1 < \cdots < r_n < s_1$.

(ii) *Let $\lambda \in \mathfrak{M}_0(I)$, $\lambda \neq 0$ and $S^-(\lambda) < \infty$. Then λ is P-extremal in $\mathscr{K}^*(L, I)$ iff $\lambda = cL^*\phi$ for some $c > 0$ and ϕ as in* (i). *In particular λ has finite support, $\int u\lambda = \sum_{i=1}^p \alpha_i u(s_i)$, where α_i/c are the coefficients obtained when expanding* (8) *along the last column. The α_i alternate in sign.*

For the G^* notation, see **[5]**. If $p = n + 1$, then ϕ is an (unnormalized) B-spline. The proof of part (i) of the theorem is not hard using known facts about Chebyshevian splines. The proof of part (ii) depends on the following two lemmas, the first of which is a result of the variation diminishing type.

LEMMA 3. *Suppose M is a kth order Chebyshevian differential operator, $\phi \in \mathfrak{M}_0^k(\boldsymbol{R})$. Let $[a, b]$ be the smallest closed interval containing* supp(ϕ). *Then $k + Z^k(\phi, (a, b)) \leqq S^-(M\phi)$.*

This lemma implies, e.g., that if ϕ is as in Theorem 1, then $S^-(L^*\phi) = p - 1$, i.e., the discrete masses of $L^*\phi$ at the s_i alternate in sign.

Using Lemma 1, the decomposition of $\lambda \in \mathscr{K}^*(L, I)$ can be dealt with in part by considering $\phi(t) = \int G(s, t)\lambda(ds)$ in $\mathfrak{M}_0^n(\boldsymbol{R})^+$. Here, the following lemma is needed.

LEMMA 4. *Suppose $\phi \in \mathfrak{M}_0^n(\boldsymbol{R})^+$ and ϕ has only zeroes of order $n - 1$ except unbounded zero-intervals. Let $f \in \mathfrak{M}_0^n(\boldsymbol{R})$ have all the zeroes of ϕ, counting multiplicities, then there is $\varepsilon > 0$ such that $\phi + af \geqq 0$ for $|a| \leqq \varepsilon$.*

3. **Applications.** We first characterize those data sets $(x_i, y_i)_{i=1}^k$ which permit a generalized convex interpolation. Cf. [1] for some results and history. Let $x_1 < x_2 < \cdots < x_k$ in the following.

LEMMA 5. *$S_0(L^*, x_1, \ldots, x_k)^+$ is spanned by its extremal elements.*

THEOREM 2. *$\psi \in S_0(L^*, x_1, \ldots, x_k)$ is extremal in $S_0(L^*, x_1, \ldots, x_k)^+$ iff $\psi = c\phi, c > 0$, with ϕ as in* (8), *$s_1, \ldots, s_p$ being any consecutive points among the x_i.*

The proof uses Lemmas 1 and 2. The foundations which allow to conclude the next theorem from the preceding results are in [1]. This result considerably improves earlier ones by T. Popoviciu and by the author.

THEOREM 3. *A set of data points $(x_i, u_0(x_i))_{i=1}^k$ admits of $u \in \mathscr{K}(L, I)$, $I \supset \{x_i\}_{i=1}^k$, such that $u(x_i) = u_0(x_i)$, $i = 1, \ldots, k$, iff $\int u_0 L^*\phi \geqq 0$ for all ϕ occurring in* (8) *(this integral is a finite sum, cf. Theorem* 1 (ii)), *with $s_1, \ldots, s_p$ any consecutive ones among the x_i, $i = 1, \ldots, k$.*

In the author's thesis [1] it is shown that, if a generalized convex interpolation exists, all such interpolations lie between two extremal ones, which are Chebyshevian L-splines with at most $k/2$ knots, on (x_1, x_k).

Next we deal with characterization of best uniform approximation by generalized convex functions. In [1] it is shown that such approximations exist on a compact interval.

THEOREM 4. *Let T be a compact Hausdorff space, $\mathscr{C}$ a cone in $\mathfrak{M}(T)$, and S^* the unit ball of the B-space $\mathfrak{M}(T) = C(T)^*$. Let $\lambda \in \mathscr{C}$, $\lambda \neq 0$. Then $\lambda/\|\lambda\|$ is an extreme point of $\mathscr{C} \cap S^*$ iff λ is P-extremal in $\mathscr{C}$.*

L. de Branges in [3] proved the "only if" part (in the form provided by Lemma 3) for the case of a w^*-closed linear subspace $\mathscr{C}$. Theorem 4 gives:

THEOREM 5. *Let $u_0 \notin \mathscr{K}(L, I)$, I a compact interval. Then there is a function ϕ as in Theorem* 1, (8) *such that if $u \notin \mathscr{K}(L, I)$, then u is a best approximation for u_0 from $\mathscr{K}(L, I)$ in the uniform norm iff*

$$\int (u - u_0)L^*\phi/\|L^*\phi\| = \|u - u_0\|_\infty; \tag{9}$$

$$\int uL^*\phi = \int \phi Lu = 0. \tag{10}$$

Equivalently, $u \in \mathscr{K}(L, I)$ *is a best approximation for* u_0 *iff*

(11) $u(s_i) = u_0(s_i) + (-1)^{p-i}\,\|u - u_0\|_\infty\ (i = 1, \ldots, p)$;

(12) *on* $[s_1, s_p]$ u *is the unique Chebyshevian L-spline with simple knots at the distinct* t_i (*zeroes of* ϕ) *satisfying* (11).

REMARKS. The alternation and uniqueness conditions in the last theorem, established for approximation by functions satisfying $Lu \geqq 0$, are seen to strongly resemble the Chebyshevian alternation and uniqueness theorem, valid for approximation by functions satisfying $Lu = 0$. Suppose $Lu = 0$ and (11) holds with $p = n + 1$, i.e. u is a best approximation to u_0 in ker L and the last sign of the error $u - u_0$ on the alternant $\{s_1, \ldots, s_p\}$ is "+". Theorem 5 implies that u is also the best approximation to u_0 in $\mathscr{K}(L, I)$.

Only for $n \leqq 2$ does the condition of Theorem 3 reduce to a finite system of linear inequalities in the $u_0(x_i)$ (if $n \leqq 2$, then (4) implies that $p = n + 1$). Our results also imply that no such system can be found for $n > 2$.

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