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EIGENFUNCTION EXPANSIONS FOR NONDENSELY DEFINED OPERATORS GENERATED BY SYMMETRIC ORDINARY DIFFERENTIAL EXPRESSIONS[1]

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1. **Nondensely defined symmetric ordinary differential operators.** This note is a sequel to [2]; the notations are the same. Let L be the formally symmetric ordinary differential operator

$$L = \sum_{k=0}^{n} p_k D^k = \sum_{k=0}^{n} (-1)^k D^k \bar{p}_k, \qquad D = \frac{d}{dx},$$

where the p_k are complex-valued functions of class C^k on an interval $a < x < b$, and $p_n(x) \neq 0$ there. In the Hilbert space $\mathfrak{H} = \mathfrak{L}^2(a, b)$ let S_0 be the closure in $\mathfrak{H}^2$ of the set of all $\{f, Lf\}$ for $f \in C_0^\infty(a, b)$, the functions in $C^\infty(a, b)$ vanishing outside compact subintervals of $a < x < b$. This S_0 in a closed densely defined symmetric operator whose adjoint has the domain $\mathfrak{D}(S_0^*)$, the set of all $f \in C^{n-1}(a, b)$ such that $f^{(n-1)}$ is absolutely continuous on each compact subinterval and $Lf \in \mathfrak{H}$. For $f \in \mathfrak{D}(S_0^*)$, $S_0^* f = Lf$. If $M_0 = S_0^* \ominus S_0$, then

$$\dim(M_0)^\pm = \dim \mathfrak{D}((M_0)^\pm) = \dim \nu(S_0^* \mp iI) = \omega^\pm,$$

say ($\nu(T)$ = null space of T). Thus $0 \leqq \omega^\pm \leqq n$, and $\dim M_0 = \omega^+ + \omega^- \leqq 2n$. Let $\mathfrak{H}_0$ be a subspace of $\mathfrak{H}$, $\dim \mathfrak{H}_0 = p < \infty$, and define the operator S, with $\mathfrak{D}(S) = \mathfrak{D}(S_0) \cap (\mathfrak{H} \ominus \mathfrak{H}_0)$, via $S \subset S_0$. We see that (2.1) of [2] is satisfied and Theorem 1 of [2] is applicable to S. If $\omega^+ = \omega^- = \omega$, which we now *assume*, then Theorem 2 of [2] is also applicable. For $u, v \in \mathfrak{D}(S_0^*)$ we have Green's formula

$$\int_y^x (\bar{v}Lu - u\overline{Lv}) = [uv](x) - [uv](y),$$

where $[uv]$ is a semibilinear form in $u, u', \ldots, u^{(n-1)}$ and $v, v', \ldots, v^{(n-1)}$. From this it follows that $[uv](x)$ tends to limits $[uv](a)$, $[uv](b)$ as x tends to a, b. Then we may write

$$\langle uv \rangle = (Lu, v) - (u, Lv) = [uv](b) - [uv](a).$$

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Thus, in Theorem 2 of [2], (ii) represents a set of boundary-integral conditions, and (iii) (or the expression for H_s) shows that both boundary and integral terms appear in the expression for the operator part of H.

2. **Eigenfunction expansions.** For any selfadjoint subspace extension $H = H_s \oplus H_\infty$ of S in $\mathfrak{H}^2$, as given in Theorem 2 of [2], we have $H_s = \int_{-\infty}^{\infty} \lambda \, dE_s(\lambda)$, where $\{E_s(\lambda)\}$ is the spectral family of projections in $\mathfrak{H} \ominus H(0)$ for H_s. We can explicitly describe the $E_s(\lambda)$ in terms of a basis for the solutions of $(L - \ell)u = \varphi \in \mathfrak{H}_0$, $\ell \in \boldsymbol{C}$. Let $\varphi_1, \ldots, \varphi_p$ be an orthornormal basis for $\mathfrak{H}_0$, and let c be fixed, $a < c < b$. Let $s_j(x, \ell)$, $a < x < b, \ell \in \boldsymbol{C}, j = 1, \ldots, n + p$, satisfy

$$(2.1)\quad \begin{aligned} &(L - \ell)s_j = 0, \quad s_j^{(k-1)}(c, \ell) = \delta_{jk}, \quad j, k = 1, \ldots, n, \\ &(L - \ell)s_{n+j} = \varphi_j, \quad s_{n+j}^{(k-1)}(c, \ell) = 0, \quad j = 1, \ldots, p, \quad k = 1, \ldots, n. \end{aligned}$$

THEOREM 1. *For any selfadjoint subspace extension $H = H_s \oplus H_\infty$ of S in $\mathfrak{H}^2$, and s_j satisfying* (2.1), *there exists an $(n + p) \times (n + p)$ matrix-valued function ρ on the real line $\boldsymbol{R}$ which is Hermitian, nondecreasing, and of bounded variation on each finite interval. Let $\Delta = \{\nu \mid \mu < \nu \leqq \lambda\}$ and $E_s(\Delta) = E_s(\lambda) - E_s(\mu)$, where λ, μ are continuity points of E_s. For $f \in C_0(a, b) \cap (\mathfrak{H} \ominus H(0))$ we have*

$$E_s(\Delta)f(x) = \int_\Delta \sum_{j,k=1}^{n+p} s_k(x, \nu)\hat{f}_j(\nu) \, d\rho_{kj}(\nu),$$

where $\hat{f}_j(\nu) = (f, s_j(\nu))$.

For vector-valued functions $\zeta = (\zeta_1, \ldots, \zeta_{n+p})$, $\eta = (\eta_1, \ldots, \eta_{n+p})$ on $\boldsymbol{R}$ we can introduce

$$(\zeta, \eta) = \int_{-\infty}^{\infty} \sum_{j,k=1}^{n+p} \zeta_j(\nu)\overline{\eta_k(\nu)} \, d\rho_{kj}(\nu).$$

Since ρ is nondecreasing, $(\zeta, \zeta) \geqq 0$ and we can define the norm $\|\zeta\| = (\zeta, \zeta)^{1/2}$, and consider the Hilbert space $\mathfrak{L}^2(\rho) = \{\zeta \mid \|\zeta\| < \infty\}$.

THEOREM 2 (EIGENFUNCTION EXPANSION). *Let H be as in Theorem* 1 *and let $f \in \mathfrak{H} \ominus H(0)$. Then $\hat{f} = (\hat{f}_1, \ldots, \hat{f}_{n+p})$ converges in norm in $\mathfrak{L}^2(\rho)$, $\|f\| = \|\hat{f}\|$, and*

$$f(x) = \int_{-\infty}^{\infty} \sum_{j,k=1}^{n+p} s_k(x, \nu)\hat{f}_j(\nu) \, d\rho_{kj}(\nu),$$

where the integral converges in norm in $\mathfrak{L}^2(a, b)$.

3. **Systems of differential operators.** The results in Theorems 1 and 2 carry over to S generated by a system of ordinary differential operators. We indicate the situation for a first order system. Let $L = P_1D + P_0$, where P_1, P_0 are $m \times m$ matrix-valued functions on $a < x < b$, with $P_1 \in C^1(a, b)$, $P_0 \in C(a, b)$, and $P_1^{-1}(x)$ existing for $a < x < b$. Thus L operates on vector-valued functions considered as $m \times 1$ matrices. We assume L is formally symmetric, i.e., $P_1^* = -P_1$, $P_0 - P_0^* = P_1'$. The relevant Hilbert space is $\mathfrak{H} = \mathfrak{L}_m^2(a, b)$, the set of all $m \times 1$ matrix-valued functions u on $a < x < b$ such that $(u, u) < \infty$. In general, for any two matrix-valued functions F, G such that G^*F is defined and can be integrated, we define $(F, G) = \int_a^b G^*F$. The domain of the operator S_0^* consists of all $f \in \mathfrak{H}$ which are absolutely continuous on each compact subinterval, and $Lf \in \mathfrak{H}$; for $f \in \mathfrak{D}(S_0^*)$, $S_0^*f = Lf$. Green's formula in this case gives

$$\int_y^x v^*Lu - (Lv)^*u = [uv](x) - [uv](y),$$

where $[uv](x) = v^*(x)P_1(x)u(x)$. The operator $S_0 \subset S_0^*$ has a domain consisting of all $f \in \mathfrak{D}(S_0^*)$ such that $\langle fg \rangle = 0$ for all $g \in \mathfrak{D}(S_0^*)$, where $\langle fg \rangle = (Lu, v) - (u, Lv)$. For $M_0 = S_0^* \ominus S_0$ we have $0 \leqq \dim M_0 \leqq 2m$. If $\mathfrak{H}_0 \subset \mathfrak{H}$, $\dim \mathfrak{H}_0 = p < \infty$, we can define S as in (2.2) of [**2**], and then (2.1) of [**2**] is valid. Theorems 1 and 2 of [**2**] can then be applied.

We describe concretely the *regular case* where a, b are finite, P'_1, P_0 are continuous on the closed interval $a \leqq x \leqq b$, and $P_1^{-1}(x)$ exists there. Then $\mathfrak{D}(S_0^*)$ is the set of all $f \in \mathfrak{H}$ which are absolutely continuous on $a \leqq x \leqq b$ and $Lf \in \mathfrak{H}$, and $\mathfrak{D}(S_0)$ is the set of those $f \in \mathfrak{D}(S_0^*)$ satisfying $f(a) = f(b) = 0$. In this case $\dim(M_0)^{\pm} = m$, and Theorem 2 of [**2**] takes the following form.

THEOREM 3. *In the regular case of a first order system L as given above, let H be a selfadjoint subspace extension of S in $\mathfrak{H}^2$, with* $\dim H(0) = s$. *Let $\varphi_1, \ldots, \varphi_p$ be an orthornormal basis for $\mathfrak{H}_0$, with $\varphi_1, \ldots, \varphi_s$ a basis for $H(0)$. Then $H = \{\{h, Lh + \varphi\}\}$ such that $h \in \mathfrak{D}(S_0^*)$, $\varphi \in \mathfrak{H}_0$, and satisfying*

(i) $(h, \Phi_0) = 0$,

(ii) $Mh(a) + Nh(b) + (h, Z) = 0$,

(iii) $\varphi = \Phi_0 c + \Phi_1[(h, \Psi) + Ch(a) + Dh(b)]$,

where Φ_0, Φ_1 are matrices with columns $\varphi_1, \ldots, \varphi_s$ and $\varphi_{s+1}, \ldots, \varphi_p$ respectively; *c, M, N, C, D are matrices of complex constants of order $s \times 1, m \times m, m \times m, (p - s) \times m, (p - s) \times m$ respectively, and*

(a) $\operatorname{rank}(M:N) = m$,

(b) $MP_1^{-1}(a)M^* - NP_1^{-1}(b)N^* = 0$,

(c) $\Psi = \Phi_1\{E + \frac{1}{2}[DP_1^{-1}(b)D^* - CP_1^{-1}(a)C^*]\}, E = E^*$,

(d) $Z = \Phi_1[DP_1^{-1}(b)N^* - CP_1^{-1}(a)M^*]$.

Conversely, if there exist M, N, C, D, E *satisfying* (a), (b) *and* Ψ, Z *are defined by* (c), (d), *then* H *defined by* (i)–(iii) *is a selfadjoint extension of* S *with* $\dim H(0) = s$. *The operator part* H_s *of* H *is*

$$H_s h = Lh - \Phi_0(Lh, \Phi_0) + \Phi_1[(h, \Psi) + Ch(a) + Dh(b)].$$

Here $(M:N)$ is an $m \times 2m$ matrix obtained by setting the columns of M next to those of N in the order indicated, and E is a $(p - s) \times (p - s)$ matrix of constants. The operator extensions H are those given by the case $s = 0$, and these properly include those studied by A. M. Krall [**3**, Theorem 5.1]. He considered the operator cases when $P_1(x) = -iI$, and $\Psi = 0$, $E = 0$, i.e., only those operators H which do not contain an integral term in the operator. (In his condition (5.5), p. 444 of [**3**], which is the analog of (d) above, $-i$ should be replaced by $+i$.)

The analogs of the expansion results, Theorems 1 and 2, are valid for the general singular case. Let $s_j(x, \ell)$, $a < x < b$, $\ell \in \boldsymbol{C}$, satisfy $(L - \ell)s_j = 0$, $s_j(c, \ell) = e_j$ for $j = 1, \ldots, m$, and $(L - \ell)s_{m+j} = \varphi_j$, $s_{m+j}(c, \ell) = 0$ for $j = 1, \ldots, p$, where $a < c < b$ and e_j is the unit vector with 1 in the jth row. Let $S(x, \ell)$ be the matrix with columns $s_1(x, \ell), \ldots, s_{m+p}(x, \ell)$.

THEOREM 4. *Let* L *be a first order system, and* $H = H_s \oplus H_\infty$ *a selfadjoint extension of* S *in* $\mathfrak{H}^2$, $\mathfrak{H} = \mathfrak{L}_m^2(a, b)$, *with* $H_s = \int_{-\infty}^{\infty} \lambda \, dE_s(\lambda)$ *in* $\mathfrak{H} \ominus H(0)$. *There exists an* $(m + p) \times (m + p)$ *matrix-valued function* ρ *on* $\boldsymbol{R}$, *which is Hermitian, nondecreasing, and of bounded variation on each finite interval. If* $\Delta = (\mu, \lambda]$, *and* μ, λ *are continuity points of* E_s, *then for* $f \in C_0(a, b) \cap (\mathfrak{H} \ominus H(0))$,

$$E_s(\Delta)f(x) = \int_\Delta S(x, v) \, d\rho(v)\hat{f}(v), \qquad \hat{f}(v) = (f, S(v)).$$

If $f \in \mathfrak{H} \ominus H(0)$, *then* $\hat{f} \in \mathfrak{L}^2(\rho)$, $\|f\| = \|\hat{f}\|$, *and*

$$f(x) = \int_{-\infty}^{\infty} S(x, v) \, d\rho(v)\hat{f}(v).$$

4. **Selfadjoint extensions in larger spaces.** In either the nth order case or first order system case, if $\dim(M_0)^+ \neq \dim(M_0)^-$ there are no selfadjoint extensions of S in $\mathfrak{H}^2$. However, there always exist such extensions in a larger space $(\mathfrak{H} \oplus \mathfrak{K})^2$, where $\mathfrak{K}$ is a Hilbert space. Let $H = H_s \oplus H_\infty$ be any such with $H_s = \int_{-\infty}^{\infty} \lambda \, dE_s(\lambda)$ on $(\mathfrak{H} \oplus \mathfrak{K}) \ominus H(0)$. Let P be the orthogonal projection of $\mathfrak{H} \oplus \mathfrak{K}$ onto $\mathfrak{H}$, and define $F_s(\lambda)f = PE_s(\lambda)f$, for $f \in \mathfrak{H} \ominus PH(0)$, $\lambda \in \boldsymbol{R}$. The proofs of Theorems 1, 2, 4 involve a

nontrivial adaptation of the method used in our earlier paper on operators [1], and we can avoid the use of the results of A. V. Štraus mentioned there. Thus we can show that these theorems are valid for any H in $(\mathfrak{H} \oplus \mathfrak{K})^2$, with E_s replaced by F_s, and $\mathfrak{H} \ominus H(0)$ replaced by $\mathfrak{H} \ominus PH(0)$. Hence it is not necessary to assume $\dim(M_0)^+ = \dim(M_0)^-$.

Detailed proofs will appear elsewhere.

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