ENERGY DECAYS LOCALLY EVEN IF TOTAL ENERGY GROWS ALGEBRAICALLY WITH TIME

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0. Introduction. In this note we announce energy decays locally like $t^{-2+\kappa}$ for solutions of hyperbolic equations, with coefficients that depend upon both position and time, in the exterior of star-shaped domains in $\mathbb{R}^3$. Here $\kappa$ is a positive constant, depending on the coefficients, defined explicitly by (8) below. Our results generalize those of Zachmanoglou [4]. He considered a class of equations with time-independent coefficients (see (12) below) and proved under hypotheses roughly analogous to ours that in $\mathbb{R}^n$ ($n \geq 3$) energy decays locally like $t^{-1+\mu}$ ($1 > \mu \geq 0$). A more important difference between the equations we consider here and those considered by Zachmanoglou in [4] is that we treat equations with solutions whose total energy may grow algebraically with $t$ while the total energy of solutions of the equations considered in [4] is conserved. In [1] we proved that the energy of solutions with bounded total energy decays locally like $t^{-2}$, but under more stringent hypotheses than those used here.

We now set the scattering problem whose solutions we investigate. Let $V$ be the exterior of a closed, bounded subset $B$ of $\mathbb{R}^3$, and let $n$ be the outward unit normal to $\partial B$. We assume that the origin lies interior to $B$ and that $\partial V \equiv \partial B$ is star-shaped:

\begin{equation}
\min_{x \in \partial V} \frac{n \cdot x}{r} \geq 0,
\end{equation}

where $x = (x_1, x_2, x_3)$ and $r^2 = x \cdot x$. Let $Q = (V \cup \partial V) \times [0, \infty)$. We use the notation $\nabla^{(3)} = (\partial / \partial x_1, \partial / \partial x_2, \partial / \partial x_3)$, $\partial = (\nabla^{(3)}, \partial / \partial t)$. We take as given a symmetric $3 \times 3$ matrix $E$, $1 \times 3$ matrices $a$ and $b$, and functions $c$ and $d$ which satisfy the following hypothesis:

(Hypothesis $H_1$) \(a\) $b$, $c$, and $E$ are in $C^1(Q)$; $a$ and $d$ are in $C^2(Q)$,
\(b\) for some $d_0 > 0$, $d(x, t) \geq d_0$ if $(x, t) \in Q$.

Let the transpose of a matrix $M$ (or $m$) be $M^T$ (or $m^T$). We suppose that $E$ is uniformly elliptic in $Q$, namely that there exist positive constants $c_0$ and...
such that
\[ C_0 \geq \max_{|E|=1} \xi E \xi^T \geq \min_{|E|=1} \xi E \xi^T \geq c_0. \]

Finally we adopt the notation
\[ A = \begin{pmatrix} E & a^T \\ a & -d \end{pmatrix}, \quad D = \begin{pmatrix} E_t & b^T \\ b & -c \end{pmatrix}, \]
and
\[ (\cdot) = \frac{(\cdot)}{\min_{|E|=1} \xi E \xi^T}. \]

We consider solutions of the mixed initial boundary value problem
\[
\begin{align*}
(2) & \quad \nabla(A(\nabla u)^T) + (b - a_t) \cdot \nabla u + \frac{1}{2}(d_t - c)u_t = 0 \quad (x \in V, t > 0), \\
(3) & \quad u(x, t) = 0 \quad (x \in \partial V, t \geq 0), \\
(4) & \quad u(x, 0) = f(x), u_t(x, 0) = g(x) \quad (x \in V),
\end{align*}
\]
where \( f \) and \( g \) are functions in \( C^1(V \cup \partial V) \) with compact support.

1. **Norms and constants.** We use the following norms:
\[ N(\cdot) \equiv \max_{V \cup \partial V} | \cdot | \quad \text{and} \quad N'(\cdot) \equiv \max_{t \geq 0} N(\cdot). \]

We shall assume that the coefficients \( c, d \) and the matrix coefficients \( a, b, E \) are such that the following are positive numbers:
\[
\begin{align*}
\alpha_1 &= 2N'(rd/d) + 2N'(ra/d) + N'(r(d_t - c)/d), \\
\alpha_2 &= 2N'(rE_r) + 2N'(r \hat{a}_r) + 4N'(r(\hat{d}_t - \hat{b})) + N'(r(d_t - \delta)) \\
&\quad + 8N(r^{-2})N'[r^4(d_t - c)/4 - \nabla^3(a_t - b)]N'(\hat{1}), \\
\alpha'_1 &= N'(tb/d) + N'(tc/d), \quad \alpha'_2 = N'(tE_t) + N'(t\hat{b}).
\end{align*}
\]

In addition to the smoothness conditions on the coefficients already imposed, this hypothesis imposes decay rates on the coefficients in (1) and on their time and space derivatives.

Let \( \Omega \) be defined by the equation
\[
\left\{ 1 + N'(\hat{d}) + N'(r^{-2})N'(\hat{1})[4N'(r^3d_t) + 2N'(r^3d_t - c)] \\
+ 4(N'(r^{-4}))^{1/2}N'(\hat{1})N'(r^2a) + 2N'(\hat{d}) \right\} = \frac{1}{2}.
\]

Again we shall assume that the coefficients of (1) are such that the above norms are well defined.

For any \( \varepsilon \in (0, 1) \), each \( T > 0 \), and some small \( \delta > 0 \), we define
\[
\begin{align*}
\mathcal{D}(T) &= \{ x \mid r \leq \varepsilon \Omega T \} \cap V, \\
\kappa &= (1 - \varepsilon)^{-1}[\max(\alpha_1, \alpha_2) + (1 + (\varepsilon \Omega)^2)\max(\alpha'_1, \alpha'_2)].
\end{align*}
\]
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\( s = \max[N'(t\dot{E}) + N'(tb), N'(tb/d) + N'(tc/d)] \),

and

\( q = -1 + \delta + s. \)

Note that if the differential equation (2) is the wave equation, then \( \kappa = 0. \)

2. Local energy decay. We make the following major hypothesis:

(Hypotheses H\(_2\)) The \( N' \)-norm of each of the following is a positive number:

\[
\begin{align*}
& r^{3+q}(E - dl), r^{4+q}d_t, r^{4+q}(d_t - c), r^{3+q}a, \\
& r^{4+q}a, r^{4+q}(a_t - b), r^{4+q}[2(d_t - c) + d_t - 2\nabla^{(3)} \cdot a], \\
& r^{5+q}(d_t - c) + 2\nabla^{(3)} \cdot (a_t - b), tE_t, r^{4+q}E_t, tb, \text{ and } t^{4+q}c.
\end{align*}
\]

Let \( \mathcal{E}_{\text{loc}}(x, t) = \frac{1}{2} \int_{\partial V} [dV^2 + \nabla^{(3)} E^T \nabla^{(3)} u] \, dx, \) and let \( \mathcal{E}(x, 0) \) be the total initial energy associated with \( u. \)

**Theorem 1.** Suppose that \( \partial V \) is star-shaped, \( E \) is uniformly strongly elliptic, Hypothesis \( H_1 \) is satisfied and the initial data \((f, g)\) in (4) are smooth and have compact support in \( V. \) Then the unique solution to Problems (2)–(4) has compact support. Moreover, suppose \( T_0 = T_0(a) \) is so large that \( \partial V \subset D(T_0), \) where \( D \) is defined by (7) and Hypothesis \( H_2 \) is satisfied by the coefficients of the differential equation (2). Then, for each \( \epsilon \in (0, 1), \) there exist positive constants \( M, K, \) and \( \kappa, \) with \( \kappa \) defined by (8), such that for \( T > T_0 \)

\[
\mathcal{E}_{\text{loc}}(x, T) \leq \frac{K\mathcal{E}(x, 0)}{T^2} \left\{ 1 + \left( \frac{T}{T_0} \right)^\epsilon e^{M/T_0} \left[ 1 - \left( \frac{T_0}{T} \right)^\epsilon + \frac{M}{(1 + \kappa)T_0} \right] \right\}.
\]

The theorem holds even if for each positive number \( p \) there exists a \( t > p \) for which the quadratic form associated with the matrix \( D \) defined by (1) fails to be negative semidefinite on \( V \cup \partial V. \) The negative semi-definiteness of \( D \) was a major hypothesis in [1].

**Corollary 1.** If the quadratic form associated with \( D \) is negative semi-definite on \( (V \cup \partial V) \times [T_0, \infty), \) then the energy decay estimate (11) holds with \( a_1' \) and \( a_2', \) which enter into the definition of \( \kappa, \) both zero and with the exponent \( q \) in Hypothesis \( H_2 \) equal to \( -1 + \delta. \)

In the case of the wave equation our decay estimate reduces to \( O(t^{-2}), \) which is the same as that obtained by C. S. Morawetz in [2]. E. C. Zachmanoglou [4] proved energy decays locally like \( t^{-1+\mu} (1 > \mu \geq 0) \) for solutions of hyperbolic equations of the form

\[
\nabla^{(3)}(E(x)(\nabla^{(3)} u)^T) - c(x)u - d(x)u_{tt} = 0
\]

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by generalizing the argument used by Morawetz in [3] in treating the wave equation. To establish a faster rate of energy decay as \( t \to \infty \) than the rate Zachmanoglou establishes in [4], we have to impose Hypothesis \( H_2 \), which is more stringent by a factor of \( r \) than his analogous conditions. But the total energy of solutions of equations of the form (2) satisfying Hypothesis \( H_2 \) may grow algebraically with \( t \).

In [1] we considered equation (2) with \( E = E(x, t) \), \( c = 0 \), and \( d = 1 \) under the hypotheses that \( E_t \leq 0 \), \( c_0 \geq 1 \), and for \( t \geq N \) and \( r \geq \varepsilon t + c \), \( |rV(E)| = O(t^{-2-\delta}) \), \( \|x(E - I)\| = O(t^{-1-\delta}) \), and \( |rE_r| = O(t^{-2-\delta}) \) for some positive \( c \) and \( \delta \).

Our methods of proof in [1] and of Theorem 1 are similar; the estimates we use to prove Theorem 1 are much sharper. Both results are based on the divergence identity

\[
\nabla[(\alpha \cdot \nabla u)(Aw) - \alpha^T(\nabla u Aw)/2 + u(\gamma A + B)w + C^T u^2/2] = [\alpha \cdot \nabla u + \gamma u](\nabla(Aw)) + (\nabla \cdot C)u^2/2 + u[1 + (\nabla \cdot \alpha)A + \nabla \cdot B] \cdot \nabla u
\]

where \( B \) is an antisymmetric \( 4 \times 4 \) matrix with \( B_{ij} = 0 \) (\( i = 1, 2, 3 \)), \( w = (\nabla u)^T \), \( B^{*i} = [-((\beta + t^2 r^{-2})x, 0)]d \), \( \alpha = (2xt, r^2 + t^2) \), \( \gamma = 2t \), \( C = (2tx dr^{-2} + \Delta d, -d - t^2 dr^{-2}) \), \( \beta \) is a solution of

\[
(\partial d_{i} + 3(\partial \beta)_{i} r^{-1}) = 3 \partial r^{-1} + t(d_i - c)r^{-1} + t^2 dr^{-2},
\]

\[
\nabla \alpha = (\alpha^T), \text{ and } \Delta d = 2t(b - a_i) - 2a + [(\beta d_i + (t^2 d_i)r^{-2})x - 2tx dr^{-2}].
\]

We integrate this identity over the space-time domain bounded by the planes \( t = T_0, t = T \) \((T > T_0)\), and the surface \( \partial V \times [T_0, T] \), apply the divergence theorem, and estimate carefully to prove Theorem 1.

REFERENCES


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