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EXTENSIONS OF C^*-ALGEBRAS, OPERATORS WITH COMPACT SELF-COMMUTATORS, AND K-HOMOLOGY[1]

BY L. G. BROWN, R. G. DOUGLAS,[2] AND P. A. FILLMORE

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1. **Introduction.** The study of a certain class of extensions of C^*-algebras is suggested by recent developments in two diverse areas of mathematics. Starting from the classical results of Weyl and von Neumann on compact perturbations of selfadjoint operators, operator theorists have become increasingly interested in operators which are normal modulo the compacts. Such operators generate extensions of the desired type and the study of these operators can be reduced to that of the corresponding extensions. Along these lines our results on extensions yield that an operator with compact self-commutator and essential spectrum not separating the plane is a normal plus a compact, and that the set of the normals plus the compacts is norm-closed.

Further, the algebras generated by pseudo-differential operators on manifolds also define such extensions as do the algebras generated by certain collections of Toeplitz and Wiener-Hopf operators. Moreover, these extensions can be used to realize "concretely" a K-homology theory outlined by Atiyah [**1**]. It seems likely that this functor will have applications in index theory and topology.

Although unanswered questions (which we describe later) remain, our results are reasonably complete, especially in regard to lower dimensional spaces which are of principal interest in the applications to operator theory. Definitions and some results appear in [**2**]. Proofs of the remaining results will appear later.

2. **The functor** Ext. Let $\mathscr{H}$ be a separable infinite dimensional complex Hilbert space, $\mathscr{L}(\mathscr{H})$ the algebra of bounded operators on $\mathscr{H}$, $\mathscr{K}$ the ideal of compact operators, $\mathfrak{A}$ the quotient (Calkin) algebra $\mathscr{L}(\mathscr{H})/\mathscr{K}$ and $\pi:\mathscr{L}(\mathscr{H}) \rightarrow \mathfrak{A}$ the quotient map. Let X be a compact metrizable space and $C(X)$ the C^*-algebra of continuous complex functions on X. We are interested in the C^*-algebra extensions of $\mathscr{K}$ by $C(X)$. We require that the algebras have identity and a further condition which guarantees

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[2] Sloan Foundation Fellow.

we have an extension for X and not for a proper closed subset of X. We let Ext(X) denote the set of equivalence classes of such extensions, where we permit arbitrary automorphisms of $\mathscr{H}$.

The elements of Ext(X) can be shown to correspond to the $*$-isomorphisms $\tau: C(X) \to \mathfrak{A}$ determined up to unitary equivalence in $\mathfrak{A}$. If τ_1 and τ_2 define elements of Ext(X), then $\tau_1 \oplus \tau_2$ (considered to have range in the Calkin algebra defined on $\mathscr{H} \oplus \mathscr{H}$) defines a binary operation making Ext(X) into a commutative semigroup. A map τ defines a trivial element of Ext(X) if there exists a $*$-isomorphism σ from $C(X)$ into $\mathscr{L}(\mathscr{H})$ such that $\tau = \pi \circ \sigma$. The trivial element τ_X exists, is unique, and acts as an identity in Ext(X). The existence of inverses follows from results set forth later in this note, and thus Ext(X) is an abelian group.

If $X \subseteq \boldsymbol{C}$, then the elements of Ext(X) are given by the normal elements of $\mathfrak{A}$ with spectrum X, or equivalently, T in $\mathscr{L}(\mathscr{H})$ defines an element of Ext(X) if $[T, T^*] = TT^* - T^*T$ is compact and the essential spectrum of T is X. This extension is trivial if and only if $T = N + K$ for some normal operator N and compact operator K.

A continuous map $f: X \to Y$ between the compact metric spaces X and Y defines a natural homomorphism $f_*: \mathrm{Ext}(X) \to \mathrm{Ext}(Y)$ and thus $X \to \mathrm{Ext}(X)$ is a covariant functor between the category of compact metrizable spaces and the category of abelian groups.

Fix an element τ of Ext(X). If g is an invertible function in $C(X)$, then $\tau(g)$ is invertible in $\mathfrak{A}$ and hence the Fredholm index ind$[\tau(g)]$ is defined and depends only on the class of g in the group $\pi^1(X) = [X, \boldsymbol{C}^*]$ of homotopy classes of maps from X to $\boldsymbol{C}^* = \boldsymbol{C}\backslash\{0\}$. Thus we obtain a homomorphism $\gamma(\tau): \pi^1(X) \to \boldsymbol{Z}$ and the mapping

$$\gamma: \mathrm{Ext}(X) \to \mathrm{Hom}(\pi^1(X), \boldsymbol{Z})$$

defines a natural transformation of covariant functors. This construction can be generalized replacing $\pi^1(X)$ by $K^1(X)$. Here as in what follows, $K^1(X)$ denotes the reduced K-group defined by

$$K^1(X) = \operatorname*{direct\,limit}_{n \to \infty} [X, GL(n, \boldsymbol{C})] = [X, GL(\infty, \boldsymbol{C})].$$

If $g: X \to GL(n, \boldsymbol{C})$, then $\gamma_\infty(\tau)(g)$ is the index of the corresponding $n \times n$ matrix of elements of $\mathfrak{A}$ which is invertible in the Calkin algebra defined on $\mathscr{H} \oplus \mathscr{H} \oplus \cdots \oplus \mathscr{H}$. Thus we obtain a natural transformation $\gamma_\infty: \mathrm{Ext}(X) \to \mathrm{Hom}(K^1(X), \boldsymbol{Z})$.

When X is the circle $\boldsymbol{T}$, the generator of Ext($\boldsymbol{T}$) can be defined in terms of Toeplitz operators [**3**]. That all extensions arise in an analogous manner follows from the existence of inverses. More precisely, for $\tau: C(X) \to \mathfrak{A}$, there exists a $*$-isomorphism $\sigma: C(X) \to \mathscr{L}(\mathscr{H}_1 \oplus \mathscr{H}_2)$ such that $\tau(\varphi)$ is π of the "Toeplitz operator" $P_{\mathscr{H}_1}\sigma(\varphi) \mid \mathscr{H}_1$.

3. Ext **as generalized homology.** If f_1 and f_2 are homotopic maps from X to Y, we show that $f_{1*} = f_{2*}$. In particular, $\text{Ext}(X) = 0$ if X is contractible. Using the homotopy invariance one can construct a long exact sequence. If SX denotes the suspension of X and S^nX the nth iterated suspension, then $\text{Ext}_{1-n}(X) = \text{Ext}(S^nX)$ ($\text{Ext}_1(X) = \text{Ext}(X)$) defines a homotopy invariant functor. For A a closed subset of X, let X/A denote the compact metric space obtained from X by identifying A to a point. Then there is a long exact sequence:

$$\text{Ext}_1(A) \to \text{Ext}_1(X) \to \text{Ext}_1(X/A) \to \text{Ext}_0(A) \to \cdots$$

Unfortunately, we do not know what to put to the left of $\text{Ext}_1(A)$ to continue exactness except in those cases in which we know that Bott periodicity holds. Note that the use of X/A in the sequence gives the strong excision property for this functor.

There is also a Mayer-Vietoris sequence for this functor. Although this follows from the long exact sequence, certain parts of it are established first and play a key role in proving the existence of inverses and homotopy invariance.

A slightly more general class of extensions is obtained from vector bundles F over X. If $\mathscr{C}_F$ denotes the C^*-algebra of continuous cross-sections of $\mathscr{L}(F)$, then the group $\text{Ext}(\mathscr{C}_F, \mathscr{K})$ of extensions of $\mathscr{K}$ by $\mathscr{C}_F$ is naturally isomorphic to $\text{Ext}(X)$, provided one uses "weak equivalence" to define $\text{Ext}(\mathscr{C}_F, \mathscr{K})$. The proof of this is closely related to the definition of γ_∞ and various other methods of relating K-theory to $\text{Ext}(X)$ which we now describe. First, $\text{Ext}_1(X)$ is a $K^0(X)$ module. Secondly, there are bilinear pairings from $K^1(X) \times \text{Ext}_1(X)$ to $\text{Ext}_0(X)$, from $K^1(X \times Y) \times \text{Ext}_1(X)$ to $K^0(Y)$ and from $K^0(X \times Y) \times \text{Ext}_0(Y)$ to $K^0(Y)$. The last was essentially defined by Atiyah in [**1**] while the second is analogous. Moreover, $\text{Ext}_0(X)$ solves the problem posed by Atiyah in [**1**]. In fact, his $\text{Ell}(X)$ can be mapped into $\text{Ext}_0(X) \oplus \boldsymbol{Z}$, and his map from $\text{Ell}(X)$ to $K_0(X)$ factors through $\text{Ext}_0(X) \oplus \boldsymbol{Z}$. Lastly, for simplicial complexes (the only spaces Atiyah considers) the map from $\text{Ext}_0(X) \oplus \boldsymbol{Z}$ to $K_0(X)$ is an isomorphism.

4. **Continuity properties of** Ext. If $CE(X)$ describes the set of $*$-homomorphisms $\tau: C(X) \to \mathfrak{A}$, then $CE(X)$ contains the maps defining the elements of $\text{Ext}(A)$ for all closed subsets A of X. Now $CE(X)$ is a complete metric space relative to the strong operator topology and it is natural to ask whether the set of maps defining trivial extensions is closed. We show for X the square or the three-cube that the set is closed, while for higher dimensional cubes the set is not closed. These results imply that the set of normal plus compact operators is closed (cf. [**3**], [**4**]), while

the collection $\{(N_1 + K_1, N_2 + K_2)\}$ is *not*, where N_1 and N_2 are commuting normal operators and K_1 and K_2 are in $\mathscr{K}$.

If A is a closed subset of X and if τ defines an element of Ext(A) which becomes trivial in Ext(U) for every closed neighborhood U of A, then τ is in the closure of the trivial elements of $CE(X)$. Further, if f is any continuous function from A to an absolute neighborhood retract (ANR), then $f_*(\tau) = 0$, and if we assume that X is an ANR, then this latter property implies the former. Thus, we are led to define PExt(X) as the set of τ in Ext(X) such that $f_*(\tau) = 0$ for all f sending X to an ANR.

The closure problem can now be divided into two parts. First, is PExt(X) $= 0$? Secondly, if $\{\tau_n\}$ and τ are isomorphisms in $CE(X)$ such that each τ_n is trivial and $\tau_n \to \tau$, then is τ trivial? The first question has an affirmative answer in the smooth case as well as in some low dimensional cases. The pairing of $K^1(X \times Y) \times \mathrm{Ext}_1(X)$ to $K^0(Y)$ and the ideas in [**1**] suffice to answer affirmatively the second question for X a simplicial complex and for any X such that PExt(X) $= 0$. In fact, an element of Ext(X) is in PExt(X) if and only if it is annihilated by this pairing. We observe also that an extension of $\mathscr{K}$ by $C(X)$ is "simultaneously quasi-diagonal" if and only if it is in PExt(X).

Next we discuss a further "continuity" property of Ext. If X is the projective limit of a sequence $\{X_n\}$ of compact metric spaces, then there is a natural map

$$P: \mathrm{Ext}(X) \to \operatorname{proj}\lim_n \mathrm{Ext}(X_n)$$

which is onto. The kernel of P is contained in PExt(X), and equals it if each X_n is an ANR. In general, Ext(X)/PExt(X) is isomorphic to proj $\lim_n$ Ext(X_n)/PExt(X_n). Hence, if one imitates the definition of Čech homology replacing homology with Ext for the simplicial complexes which occur (as the nerves of the finite open covers of X), then one obtains Ext(X)/PExt(X) and thus Ext(X)/PExt(X) is "(reduced) Čech K-homology". Since Čech K-homology is not exact, PExt(X) is not always 0.

5. **Bott periodicity and a topology for** Ext. A Bott periodicity map from $\mathrm{Ext}_{-1}(X)$ to $\mathrm{Ext}_1(X)$ can be defined using the module action of $K^0(X)$ on $\mathrm{Ext}_1(X)$. This map is always one-to-one but we do not know if it is always onto. The work of Venugopalkrishna [**5**] shows it is onto for spheres and standard techniques yield that it is onto for simplicial complexes. Also Bott periodicity does define an isomorphism between $\mathrm{Ext}_{-1}(X)/\mathrm{PExt}_{-1}(X)$ and $\mathrm{Ext}_1(X)/\mathrm{PExt}_1(X)$. Whenever the map from $\mathrm{Ext}_{-1}(A)$ to $\mathrm{Ext}_1(A)$ is onto, it is clear how to define a map from $\mathrm{Ext}_0(X/A)$ to $\mathrm{Ext}_1(A)$ which gives a cyclic exact sequence with six terms.

It is often possible to compute Ext(X) when X is a complex. This group is finitely generated and γ_∞ has finite kernel and is onto. Using this, one can topologize Ext(X) in a natural way. For simplicial complexes Ext(X) is given the discrete topology and in the general case one represents X as a projective limit of spaces homotopic to simplicial complexes and uses the projective limit topology. Now Ext(X) may not be a Hausdorff group, since the closure of the identity is PExt(X). The kernel of γ_∞ can be shown to be the maximum compact subgroup of Ext(X). Another method of defining the same topology on Ext(X) is to consider the quotient topology on the isomorphisms in $CE(X)$.

6. **Concluding remarks.** Certain more refined results are possible for low dimensional X.

(a) If X is a subset of R^3 (in particular, if dim $X \leqq 1$), then $\gamma: \mathrm{Ext}_1(X) \to \mathrm{Hom}(\pi^1(X), \boldsymbol{Z})$ is a bijection ($\pi^1(X) = K^1(X)$ in this case). The real projective plane gives an example showing that the dimensionality condition is sharp. As a corollary to this result we obtain that PExt(X) $= 0$ for X a subset of R^3.

(b) If X is a subset of R^3 or dim $X \leqq 2$, then

$$\mathrm{Ext}_1(X)/\mathrm{PExt}_1(X) \cong \check{H}_1(X).$$

(c) If X is a subset of R^4 or dim $X \leqq 3$, then

$$\mathrm{Ext}_0(X)/\mathrm{PExt}_0(X) \cong \check{H}_0(X) \oplus \check{H}_2(X).$$

The maps of $\mathrm{Ext}_n(X)$ to the Čech homology groups in (b) and (c) commute with the maps of the long exact sequence when applicable. This implies in case (a) that the Čech homology sequence is exact at the appropriate groups and in cases (b) and (c) that exactness of the Čech homology sequence at these groups is a necessary condition for $\mathrm{PExt}_0(X)$ and $\mathrm{PExt}_1(X)$ to be trivial. This observation enables one to produce a two dimensional space X such that PExt(X) $\neq 0$.

Result (a) yields the following classification theorem. If the operators S and T have compact self-commutators and the same essential spectrum X, then S is unitarily equivalent to a compact perturbation of T if and only if $\mathrm{ind}(S - \lambda) = \mathrm{ind}(T - \lambda)$ for λ not in X. This in turn yields the following canonical forms for such operators: (i) T is a normal plus a compact if $\mathrm{ind}(T - \lambda) = 0$ for λ not in X, (ii) T is a subnormal plus a compact if $\mathrm{ind}(T - \lambda) \leqq 0$ for λ not in X, and (iii) T is the direct sum of a subnormal and the adjoint of a subnormal plus a compact in general.

One can view the results of this paper as a special case of classifying C^*-algebras with a given dual.

A generalization of Ext to rational coefficients can be obtained by replacing $\mathscr{K}$ by a C^*-algebra obtained as the direct limit of the compact operators.

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DEPARTMENT OF MATHEMATICS, SUNY AT STONY BROOK, STONY BROOK, NEW YORK 11790 (Current address of L. Brown and R. Douglas)

DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, BLOOMINGTON, INDIANA 47401

DEPARTMENT OF MATHEMATICS, DALHOUSIE UNIVERSITY, HALIFAX, NOVA SCOTIA, CANADA (Current address of P. Fillmore)