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TOPOLOGICAL INVARIANCE OF CERTAIN COMBINATORIAL CHARACTERISTIC CLASSES

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$(X, \partial X)$ denotes a finite polyhedral pair which is a rational homology manifold pair. σ denotes an additive invariant associated to nonsingular quadratic forms over the rationals, e.g., the index the discriminate. In this note we prove what the title says, for certain combinatorial invariants $\gamma(X, \sigma)$ associated to X.

The classes $\gamma(X, \sigma)$, which generalize the combinatorial Pontrjagin classes, occur as one of the two following types:

(a) If the additive invariant σ is the index, then

$$\gamma(X, \sigma) \in K_x^{\overline{G/\mathrm{TOP}}}(X, \partial X)^1$$

where $x = \dim(X)$; for X a PL manifold, $\gamma(X, \sigma)$ localized away from 2 coincides with the KO_* orientation class for X defined in [**8**]; and $\gamma(X, \sigma) \otimes_z Q$ is equivalent to the PL Pontrjagin classes.

(b) If σ has finite exponent, then $\gamma(X, \sigma) \in \sum_i H_{4i+x}((X, \partial X), Z_4)$. Rational manifolds are the only possible fixed point sets of PL actions of groups of prime order on manifolds [**6**]. If X is the fixed point set of such an action then the bocksteins of certain of the exponent four classes $\gamma(X, \sigma)$ must vanish [**3**].

For any closed, rational homology manifold Y, $\sigma(Y)$ will denote the evaluation of σ on the mid-dimensional intersection pairing of $H_*(Y, Q)$. Note that $\sigma(Y) = 0$ if $\dim(Y) \neq 0$ (4). Let $\{P\}$ denote the set of subpolyhedra in $X \times D^L$ (L = large) which have either linear normal bundles or linear normal bundles with "Z_q-type" singularities [**4**], [**7**]. It is an important theorem that the classes $\gamma(X, \sigma)$ can be identified with the geometric construction $\{P\} \to \{\sigma(P)\}$ (see [**7**], [**8**], and compare with [**3**]).

Let $(X, \partial X)$, $(X', \partial X')$ denote finite polyhedral pairs which are rational homology manifold pairs.

THEOREM. *If $f:(X, \partial X) \to (X', \partial X')$ is a topological homeomorphism then $f_*(\gamma(X, \sigma)) = \gamma(X', \sigma)$.*

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[1] $\overline{G/\mathrm{TOP}}$ is the torsion free (in homotopy) H-space factor of G/TOP, with respect to the "characteristic variety" H-space structure for G/TOP [7]. $K_*^{\overline{G/\mathrm{TOP}}}(\ )$ denotes the homology theory having $\overline{G/\mathrm{TOP}}$ as its zeroth loop spectrum.

PROOF. It will suffice to consider $\sigma(P)$ for those polyhedra $P \subset X \times D^L$ which have $P \times D^m$ for a regular neighborhood.

Following Novikov [4], let $T^{m-1} \times I \subset D^m - \partial D^m$ denote the standard embedding of the $(m-1)$-torus, crossed with the unit interval, into the open m-ball. Consider the restriction

$$\bar{f}: P \times T^{m-1} \times I^0 \to f(P \times T^{m-1} \times I^0).$$

$f(P \times T^{m-1} \times I^0)$ has an "end" E in the finite CW category, because $P \times T^{m-1} \times I^0$ does and $f(P \times T^{m-1} \times I^0)$ is properly homotopically equivalent to $P \times T^{m-1} \times I^0$ under $\bar{f}, \bar{f}^{-1}$ (see [5], [9]).[2]

By adding the end E to $f(P \times T^{m-1} \times I^0)$, *outside* a compact rational homology manifold neighborhood R for $f(P \times T^{m-1} \times 1/2)$ in $f(P \times T^{m-1} \times I^0)$, a CW complex triple $(Y, \partial_+ Y, \partial_- Y)$ is constructed satisfying

(i) $(Y, \partial_+ Y, \partial_- Y)$ is homotopy equivalent to $P \times T^{m-1} \times (I, 0, 1)$.

(ii) R^0 is contained in Y as an open set, and the orientation class for $(Y, \partial Y)$ restricts on $(R, \partial R)$ to the orientation class for $(R, \partial R)$.

Finally by putting the composition map

$$(Y, \partial_\pm Y) \sim P \times T^{m-1} \times (I, \partial_\pm I) \xrightarrow{P_2 \times P_3} T^{m-1} \times (I, \partial_\pm I)$$

in transverse position to $T^{m-1} \times 1/2$ (see (ii) above), we obtain a "cobordism" W from $P \times T^{m-1}$ to a polyhedron L which is a PL collared subset of $f(P \times T^{m-1} \times I^0)$. Note that there is a canonical map $h: W \to T^{m-1}$, and that $\gamma(X, \sigma)$ is computed "on P" as $\sigma(h^{-1}_{|\partial_- W}(t_0))$, where $t_0 \in T^{m-1}$. The corresponding computation for $\gamma(X', \sigma)$ is $\sigma(h^{-1}_{|\partial_+ W}(t_0))$.

To complete the proof of the theorem it must be shown that $\sigma(h^{-1}_{\partial_+ W}(t_0)) = \sigma(h^{-1}_{\partial_- W}(T_0))$. We do this by constructing a rational Poincaré duality cobordism from $h^{-1}_{|\partial_+ W}(t_0)$ to $h^{-1}_{|\partial_- W}(t_0)$. First note that W is actually a Poincaré cobordism with respect to the coefficients $Q(\pi_1(T^{m-1}))$ (see (i), (ii) above). Use the PL rational homology manifolds structures of $\partial_\pm W$ to put $h_{|\partial_\pm W}$ in transverse position, simplex by simplex to the sequence $t_0 \subset T^1 \subset T^2 \subset T^3 \subset \cdots \subset T^{m-2} \subset T^{m-1}$. There is one surgery obstruction, $S(h, \partial h)$, to extending this sequential transversality to all of h *in the category* of codimension one nested spaces which are Poincaré with respect to the nested coefficients $Q \subset Q(\pi_1(T^1)) \subset \cdots \subset Q(\pi_1(T^{m-1}))$ (see §7.11 of [2]).

It only remains to see $S(h, \partial h) = 0$. It is helpful to consider $S(h, \partial h)$ in the following simple (but, by the constructions of [2], universally typical) case. M, N are two, compact, differentiable manifolds with dimensions

[2] To construct ends in the finite CW category, replace the handlebody techniques used in [5], by the cellular techniques of [9].

$\gg m$, having boundary components $\partial_i M$, $\partial_j N$. Let the maps

$$\partial_0 M \subset M \xrightarrow{h_M} T^{m-1} \xleftarrow{h_N} N \supset \partial_0 N$$

induce isomorphisms of fundamental groups. $g:\partial_0 M \to \partial_0 N$ is a homology equivalence with respect to the coefficients $Q(\pi_1(T^{m-1}))$, and g commutes with h_M, h_N. Let $h: W \to T^{m-1}$ equal the union along g of h_M and h_N. A transversality of $h_{|\partial W}$ to $t_0 \subset T^1 \subset T^2 \subset \cdots \subset T^{m-1}$ extends to all of h if $g:\partial_0 M \to \partial_0 N$ can be made transversal to

$$h_{\partial_0 N}^{-1}(t_0 \subset T^1 \subset \cdots \subset T^{m-1})$$

in such a way that

$$g:g^{-1}(h_{\partial_0 N}(t_0 \subset T^1 \subset \cdots \subset T^{m-1})) \to h_{\partial_0 N}^{-1}(t_0 \subset T^1 \subset \cdots \subset T^{m-1})$$

is a homology equivalence with respect to the nested coefficients $Q \subset Q(\pi_1(T^1)) \subset \cdots \subset Q(\pi_1(T^{m-1}))$. This is precisely what the "rational form" of the Farrel-Hsiang splitting theorem allows [1]. It might be necessary to first vary $g:\partial_0 M \to \partial_0 N$ through a cobordism which is a homological H-cobordism with respect to the coefficients $Q(\pi_1(T^{m-1}))$ before achieving the desired transversality of g. But such a variation is allowed in the argument of the previous paragraph. Q.E.D.

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