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PERTURBATION OF SEMIGROUPS ON LOCALLY CONVEX SPACES[1]

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1. **Introduction.** The results announced here deal with the perturbation of a one-parameter semigroup of operators on a locally convex space.

Historically the study of the semigroup $\{T_t : t \geqq 0\}$ with infinitesimal generator A has depended heavily on the relation between the semigroup and the resolvent of the generator $(\lambda - A)^{-1} = R(\lambda, A)$ given by the Laplace transform

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t} T_t \, dt. \tag{1}$$

This relation is central to the study of semigroups on Banach spaces [**2**, pp. 360–364], equicontinuous semigroups on locally convex spaces [**9**, pp. 246–248], and distribution semigroups [**5**]. However, if T_t is a semigroup on a locally convex space, and T_t is not equicontinuous, then (1) may diverge for every complex λ and A will have no resolvents. In this case it may be possible to devise a *generalized resolvent* (see [**4**] and [**7**]) to replace the classical resolvent.

It is customary to associate the semigroup T_t with the homogeneous evolution equation (Cauchy problem)

$$(d/dt)f = Af \qquad \text{with } f(0) = u \tag{2}$$

having solution $f(t) = T_t u$. Here we take a different point of view and consider the inhomogeneous equation

$$(d/dt)f = Af + g \qquad \text{with } f(0) = 0. \tag{3}$$

Phillips [**7**, §6] discusses (3) on Banach spaces and concludes that if A is the generator of a strongly continuous semigroup, (3) has the solution

$$f(t) = \int_0^t T_{t-s}\, g(s)\, ds. \tag{4}$$

Kato [**3**, pp. 486, 487] proves the uniqueness of the solution.

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The uniqueness of the solution to (3) is equivalent to the existence of $(d/dt - A)^{-1}$ as an operator on *certain* vector valued functions. The operator $(d/dt - A)^{-1}$ is formally just the inverse Laplace transform of the resolvent $(\lambda - A)^{-1}$, and $(d/dt - A)^{-1}$ will be extended to define the generalized resolvent used here.

Phillips [7, §6] points out the similarity between (3) and the perturbed homogeneous equation $(d/dt)f = Af + Bf$, and uses the solution to the former to derive the perturbed semigroup. These methods suggest the potential usefulness of the above generalized resolvent in solving the perturbation problem.

2. **The generalized resolvent.** Let E be a sequentially complete Hausdorff locally convex topological vector space. A collection Γ of continuous seminorms on E generating the topology of E is called a *calibration* for E. The calibration consisting of all continuous seminorms will be denoted by Λ. A semigroup on E is a collection of operators $\{T_t : t \geqq 0\} \subset \mathscr{L}(E)$ satisfying $T_t T_s = T_{t+s}$ and $T_0 = I$ (the identity operator). The semigroup is said to be of class C_0 if the map $t \to T_t$ is continuous in the strong operator topology; it is locally equicontinuous if, for some $a > 0$, $\{T_t : 0 \leqq t \leqq a\}$ is equicontinuous. The infinitesimal generator A of the semigroup T_t is defined by

$$Au = \lim\{t^{-1}(T_t u - u) : t \to 0\}$$

with domain $\mathscr{D}(A)$ consisting of those vectors u for which the limit exists.

REMARKS. If T_t is locally equicontinuous then $\{T_t : 0 \leqq t \leqq a'\}$ is equicontinuous for every $a' > 0$. If T_t is of class C_0 and E is barrelled then T_t is locally equicontinuous [4].

The operators $(\lambda - A)$ and $R(\lambda, A)$ act on the space E, but the operator $(d/dt - A)$ acts on the space of E valued functions of the real line. In order to obtain a vector space of functions in which (3) has unique solutions we must consider only functions vanishing at 0. We denote by $\mathfrak{X}_a$ the space of all continuous E valued functions on the interval $[0, a]$ that vanish at 0. When there is no chance of confusion $\mathfrak{X}$ will be used instead of $\mathfrak{X}_a$. Two topologies will be needed on $\mathfrak{X}$.

DEFINITION 1. (a) Suppose Γ is a calibration for E and $p \in \Gamma$. Let $p_\infty(f) = \sup\{p(f(t)) : t \in [0, a]\}$. Let $\Gamma^\infty = \{p_\infty : p \in \Gamma\}$. Γ^∞ calibrates the topology $\mathscr{T}^\infty$ of uniform convergence on $[0, a]$.

(b) Let $p_1(f) = \int_0^a p(f(t))\, dt$. Let $\Gamma^1 = \{p_1 : p \in \Gamma\}$. Γ^1 calibrates the topology $\mathscr{T}^1$ of L^1 ($[0, a]$) convergence.

REMARK. The topologies $\mathscr{T}^\infty$ and $\mathscr{T}^1$ are independent of the choice of the calibration Γ.

The operators d/dt, A, and $(d/dt - A)^{-1}$ must all be considered as acting on the space $\mathfrak{X}$.

DEFINITION 2. (a) The operation of differentiation on $\mathfrak{X}$ is denoted by $\mathfrak{D}$. The domain of $\mathfrak{D}$ is

$$\mathscr{D}(\mathfrak{D}) = \{f \in \mathfrak{X} : (d/dt)f \in \mathfrak{X}\} \quad \text{and} \quad \mathfrak{D}f = (d/dt)f.$$

(b) The operation of pointwise action by an operator A (or B) is denoted by $\mathfrak{A}$ (or $\mathfrak{B}$), $\mathscr{D}(\mathfrak{A}) = \{f \in \mathfrak{X} : f(t) \in \mathscr{D}(A)$ for $t \in [0, a]$ and the map $t \to Af(t)$ is continuous$\}$. $(\mathfrak{A}f)(t) = A(f(t))$.

Let Φ denote the set of real valued C^∞ functions of the real line with supports contained in $[0, a]$.

The basic relationship between the solutions to (2) and (3) can now be stated. In order for (2) to have solutions for sufficiently many initial values u, it is necessary and sufficient that (3) has solutions f for sufficiently many g in $\mathfrak{X}$, and for each $p \in \Lambda$ there is a $q \in \Lambda$ such that $p_\infty(f) \leqq q_1(g)$.

THEOREM 1. *Let E be a sequentially complete locally convex space. An operator A is the infinitesimal generator of a locally equicontinuous semigroup T_t of class C_0 if and only if:*

(I) *A is closed and densely defined.*

(II) *There exists an operator $\mathfrak{R} \in \mathscr{L}(\mathfrak{X}, \mathscr{T}^\infty)$ satisfying:*

(a) *$\mathfrak{R}(\mathfrak{D} - \mathfrak{A})f = f$ for all $f \in \mathscr{D}(\mathfrak{D} - \mathfrak{A})$.*

(b) *If $f \in \mathscr{D}(\mathfrak{D})$ then $\mathfrak{R}f \in \mathscr{D}(\mathfrak{D})$ and $\mathfrak{D}\mathfrak{R}f = \mathfrak{R}\mathfrak{D}f$.*

(c) *If $f \in \mathscr{D}(\mathfrak{A})$ then $\mathfrak{R}f \in \mathscr{D}(\mathfrak{A})$ and $\mathfrak{A}\mathfrak{R}f = \mathfrak{R}\mathfrak{A}f$.*

(d) *For each $p \in \Lambda$ there is a $q \in \Lambda$ such that $p_\infty(\mathfrak{R}f) \leqq q_1(f)$ for all $f \in \mathfrak{X}$.*

The operator $\mathfrak{R}$ is the generalized resolvent and is given by

$$(5) \qquad (\mathfrak{R}g)(t) = \int_0^t T_{t-s}\, g(s)\, ds$$

just as solutions to (3) are given by (4). The proof that $\mathfrak{R}$ as defined by (5) satisfies (II) is straightforward.

The converse is proved by taking a sequence $\{\phi_n\} \subset \Phi$ that satisfies

(i) $\phi_n(t) \geqq 0$,

(ii) $\operatorname{supp}(\phi_n) \subset [0, 1/n]$,

(iii) $\int_{-\infty}^{\infty} \phi_n(t)\, dt = 1$.

Such a sequence is an approximate identity. For $\phi \in \Phi$, $\phi \otimes u \in \mathfrak{X}$ is defined by $\phi \otimes u(t) = \phi(t)u$ for every $u \in E$. The semigroup T_t is then defined on $(0, a]$ by $T_t u = \lim\{(\mathfrak{R}\phi_n \otimes u)(t) : n \to \infty\}$.

Conditions (II)(a), (b), (c) imply T_t is a semigroup. Condition (II) (d) guarantees that the limit exists and is continuous and equicontinuous on $(0, a)$. Furthermore $T_t u \to u$ as $t \to 0$. Hence T_t can be extended to $[0, \infty)$ giving a locally equicontinuous semigroup of class C_0.

Perturbation problems require the study of operators A that need not be closed. A more technical version of Theorem 1 can be proved for densely defined operators A that guarantees the closure $\mathrm{Cl}(A)$ exists and is the generator of a locally equicontinuous semigroup of class C_0.

3. **Perturbation by relatively bounded operators.** Phillips [7, §3] studies the perturbation of a semigroup on a Banach space by the addition of a continuous operator B to the generator A. He shows that the resolvent of the perturbed generator $A + B$ is given by

$$R(\lambda, A + B) = \sum \{R(\lambda, A)[BR(\lambda, A)]^n : n = 0, 1, \ldots\}. \tag{6}$$

Miyadera [**6**] considers the larger class of perturbing operators B satisfying:

(a) $\mathscr{D}(B) \supset \mathscr{D}(A)$ and $BR(\lambda, A) \in \mathscr{L}(E)$ for some λ.

(b) For some $K > 0$, $\int_0^1 \|BT_t u\| \, dt \leqq K \|u\|$ for all $u \in \mathscr{D}(A)$.

He shows that if B satisfies (a) and (b), then for $|z|$ sufficiently small $A + zB$ is the generator of a semigroup. The theory was further generalized by Babalola [**1**] to quasi-equicontinuous semigroups on locally convex spaces with some additional restrictions on the perturbing operator B.

Miyadera's condition (a) is equivalent to the following relative boundedness condition:

(a′) $\mathscr{D}(B) \supset \mathscr{D}(A)$ and there exist nonnegative constants L and M satisfying $\|Bu\| \leqq L \|u\| + M \|Au\|$ for all $u \in \mathscr{D}(A)$.

Definition 3. Suppose A and B are operators on a locally convex space E. B is called *relatively bounded* with respect to A if:

(i) $\mathscr{D}(B) \supset \mathscr{D}(A)$.

(ii) For each $p \in \Lambda$ there is a $q \in \Lambda$ satisfying $p(Bu) \leqq q(u) + q(Au)$ for all $u \in \mathscr{D}(A)$.

Theorem 2. *Let E be a sequentially complete locally convex space. Let T_t be a locally equicontinuous semigroup of class C_0 with generator A. Let B satisfy:*

(I) B is relatively bounded with respect to A.

(II) There is a calibration Γ for E and a $K > 0$ such that $\int_0^a p(BT_t u) \, dt \leqq Kp(u)$ for all $u \in \mathscr{D}(A)$, $p \in \Gamma$.

If $|z| < K^{-1}$, then $A + zB$ is closable and $\mathrm{Cl}(A + zB)$ *is the generator of a locally equicontinuous semigroup of class $C_0 T_t(A + zB)$.*

The theorem is proved by defining a perturbed generalized resolvent $\mathfrak{R}(A + zB)$ satisfying condition (II) of Theorem 1. Equation (6) suggests the defining formula

$$\mathfrak{R}(A + zB) = \sum \{\mathfrak{R}[zB\mathfrak{R}]^n : n = 0, 1, \ldots\}$$

that is used.

We will sketch a proof of the fact that $\mathfrak{R}(A + zB)$ satisfies condition (II) of Theorem 1 under the assumption that A and B are continuous operators on E.

REMARK. Even in the simplified case of continuous A and B the result is still of interest. In fact, one reason for studying operators on locally convex spaces is because locally convex topologies can be chosen to make operators continuous.

PROOF SKETCH FOR THEOREM 2. From Theorem 1 for $\mathfrak{R}$ we have $\mathfrak{A}\mathfrak{R}f = \mathfrak{D}\mathfrak{R}f - f$ and $\mathfrak{R}\mathfrak{A}f = \mathfrak{R}\mathfrak{D}f - f$ for every $f \in \mathscr{D}(\mathfrak{D})$. It is also clear that $\mathscr{D}(\mathfrak{D})$ is invariant under $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{R}$. We now show that $\mathfrak{R}(A + zB)$ satisfies condition (II) of Theorem 1. For (II)(a)

$$\begin{aligned}\mathfrak{R}(A + zB)(\mathfrak{D} - (\mathfrak{A} + z\mathfrak{B}))f &= \sum (z\mathfrak{R}\mathfrak{B})^n\mathfrak{R}(\mathfrak{D} - \mathfrak{A} - z\mathfrak{B})f \\ &= \sum (z\mathfrak{R}\mathfrak{B})^n f - \sum (z\mathfrak{R}\mathfrak{B})^n(z\mathfrak{R}\mathfrak{B})f = f\end{aligned}$$

for $f \in \mathscr{D}(\mathfrak{D})$. Condition (II)(b) is immediate. For (II)(c)

$$\mathfrak{A}\mathfrak{R}(A + zB)f = \mathfrak{D}\mathfrak{R}(A + zB)f - f - z\mathfrak{B}\mathfrak{R}(A + zB)f$$

or

$$(\mathfrak{A} + z\mathfrak{B})\mathfrak{R}(A + zB)f = \mathfrak{D}\mathfrak{R}(A + zB)f - f.$$

Similarly

$$\mathfrak{R}(A + zB)\mathfrak{A}f = \mathfrak{R}(A + zB)\mathfrak{D}f - f - z\mathfrak{R}(A + zB)\mathfrak{B}f$$

or

$$\mathfrak{R}(A + zB)(\mathfrak{A} + z\mathfrak{B})f = \mathfrak{R}(A + zB)\mathfrak{D}f - f$$

for $f \in \mathscr{D}(\mathfrak{D})$. Condition (II)(c) follows by extension. A straightforward calculation shows condition (II)(d) follows from hypothesis II and equation (5).

4. **The perturbed semigroup.** Phillips [7, §3] gives the following formula for the perturbed semigroup:

$$T_t(A + B) = \sum \{S_n(t) : n = 0, 1, \ldots\} \tag{7}$$

where $S_0(t) = T_t$ and $S_n(t) = \int_0^t T_s B S_{n-1}(t - s)\, ds$. A modified version of this formula is valid in the present setting.

The operators $\bar{S}_n(t)$ for $t \in [0, a]$ are defined inductively. Let $\bar{S}_{-1}(t) = 0$, $\bar{S}_0(t) = T_t$.

LEMMA 3. *Let E, T_t, A, B, Γ, and K be as in Theorem 2. Assume $\{\bar{S}_n(t) : t \in [0, a], 0 \leqq n \leqq N\} \subset \mathscr{L}(E)$ satisfies:*

(I_n) *For each $p \in \Gamma$ there is a $q \in \Gamma$ (depending on p but independent of n) such that $p(\bar{S}_n(t)u) \leqq K^n q(u)$ for all $u \in E$.*

(II_n) *$\bar{S}_n(t)$ is a continuous function of t in the strong operator topology.*

(III_n) *$d/dt\bar{S}_n(t)u = \bar{S}_n(t)Au + \bar{S}_{n-1}(t)Bu$ for all $u \in \mathscr{D}(A)$, for each*

$0 \leqq n \leqq N$. *Then* $S_{N+1}(t)$ *defined by*

$$S_{N+1}(t)u = \int_0^t \bar{S}_N(s)BT_{t-s}u\, ds \qquad \text{for all } u \in \mathscr{D}(A)$$

can be extended to an operator $\bar{S}_{N+1}(t)$ *satisfying* (I_{N+1}), (II_{N+1}), *and* (III_{N+1}).

THEOREM 4. *If* E, T_t, A, B, *and* K *are as in Theorem* 2 *and if* $|z| < K^{-1}$ *the perturbed semigroup* $T_t(A + zB)$ *is given by*

$$T_t(A + zB) = \sum \{z^n \bar{S}_n(t) : n = 0, 1, \ldots\} \quad \text{for } t \in [0, a].$$

REMARK. It follows easily from Theorem 4 that the perturbation is analytic. That is, for $|z| < K^{-1}$, the map $z \to T_t(A + zB)$ is holomorphic into $\mathscr{L}_b(E)$ (the continuous operators with the topology of uniform convergence on bounded sets).

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