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RATIONAL APPROXIMATION TO CERTAIN ENTIRE FUNCTIONS IN $[0, +\infty)$

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Recently Chebyshev rational approximation to certain entire functions on $[0, \infty)$ has attracted the wide attention of many mathematicians. In this respect the papers [1]–[6] are worth mentioning. All the above quoted papers have been devoted to entire functions of finite order. In this paper we announce results for entire functions of infinite order along with some other results for functions of finite and zero orders. We obtain much more precise information than in the earlier work [3] for entire functions of zero order. We also give an example of an entire function of zero order which shows how closely one can approximate entire functions of very small growth. The detailed proofs of these results along with some other results will appear elsewhere.

For any nonnegative integer n, let π_n denote the collection of all real polynomials of degree at most n. Let

$$\lambda_{0,n} \equiv \inf_{p_n \in \pi_n} \left\| \frac{1}{f(x)} - \frac{1}{P_n(x)} \right\|_{0 \leqq x < \infty}.$$

Now we state our results:

THEOREM 1. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be an entire function with* $a_0 > 0$ *and* $a_n \geqq 0$ $(n \geqq 1)$. *Then for any* $\varepsilon > 0$, *there are infinitely many values of* m, *such that*

$$\lambda_{0,m} \leqq e^{-m/(\log m)^{1+\varepsilon}}. \tag{1}$$

THEOREM 2. *Let* $f(z)$ *be any entire function of infinite order with nonnegative coefficients. Then for any* $\varepsilon > 0$, *there exist infinitely many values of* m *such that*

$$\lambda_{0,m} \geqq e^{-\varepsilon m}. \tag{2}$$

THEOREM 3. *Let* $f(z) = \sum_0^{\infty} a_n z^n$ *be an entire function of finite order* ρ *with* $a_0 > 0$ *and* $a_n \geqq 0$ $(n \geqq 1)$. *Then for any* $\varepsilon > 0$,

$$\liminf_{n \to \infty} (\lambda_{0,n})^{(\rho+\varepsilon)/n} \leqq 0.8 \tag{3}$$

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THEOREM 4. *Let* $f(z) = \sum_0^\infty a_n z^n$, $a_0 > 0$, $a_n \geqq 0$ $(n \geqq 1)$. *Let*

$$M(r) = \max_{|z|=r} |f(z)|, \quad \textit{and} \quad 1 \leqq \limsup_{r\to\infty} \frac{\log\log M(r)}{\log\log r} = \Lambda < 2.$$

Then for any $\varepsilon > 0$,

$$\liminf_{n\to\infty} (\lambda_{0,n})^{(1/n)(\Lambda - 1 + \varepsilon) - 1} = 0. \tag{4}$$

THEOREM 5. *Let* $f(x) = \sum_0^\infty q^{kj} x^j$ *where* $0 < q < 1$ *and* $2 \leqq k < \infty$. *Then*

$$q \leqq \liminf_{n\to\infty} (\lambda_{0,n})^{1/n^k} \leqq \limsup_{n\to\infty} (\lambda_{0,n})^{1/n^k} \leqq q^{(1-2^{1-k})}. \tag{5}$$

THEOREM 6. *Let* $f(x)$ *be a real-valued continuous function* (*not* $\not\equiv 0$) *on on any finite interval* $[0, b]$ *and assume that there exist a sequence of real polynomials* $\{P_n(x)\}_0^\infty$ *with* $p_n \in \pi_n$ *for each* $n \geqq 0$, *and a real number* $R > 1$ *such that*

$$\limsup_{n\to\infty} \left\{ \left\| \frac{1}{f(x)} - \frac{1}{P_n(x)} \right\| \right\}^{1/n^\alpha} \leqq \frac{1}{R} < 1, \quad \textit{for any } 0 < \alpha < 1. \tag{6}$$

Then $f(x)$ *is infinitely differentiable on* $[0, b]$.

ADDED IN PROOF. Quite recently we have obtained some interesting results involving the lower order of an entire function. We have also constructed an example of an entire function of infinite order which can be approximated as close as we please for a sequence of values of "n". The statements of these and some other results have been submitted to the Bull. Amer. Math. Soc.

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