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THE HOMEOMORPHISM PROBLEM FOR S^3

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1. **Introduction.** Let M be a closed, orientable 3-manifold which is defined by a Heegaard splitting of genus g. Each such Heegaard splitting may be associated with a self-homeomorphism of a closed, orientable surface of genus g (the surface homeomorphism is used to define a pasting map) and it will be assumed that this surface homeomorphism is given as a product of standard twist maps [3] on the surface. We assert:

THEOREM 1. *If M is defined by a Heegaard splitting of genus $\leqq 2$, then an effective algorithm exists to decide whether M is topologically equivalent to the 3-sphere S^3. This algorithm also applies to a proper subset of all Heegaard splittings of genus >2.*

This result is of interest because it had not been known whether such an algorithm was possible for $g \geqq 2$, and also because the algorithm has a possible application in testing candidates for a counterexample to the Poincaré conjecture.

In this note we will describe the algorithm, and sketch a brief proof. Related results about the connections between representations of 3-manifolds as Heegaard splittings, and as branched coverings of S^3, are summarized at the end of this paper. A detailed report will appear in another journal.

2. **The algorithm.** Let X_g be a handlebody of genus $g \geqq 0$ which is imbedded in Euclidean 3-space as illustrated in Figure 1. Let X_g' be a

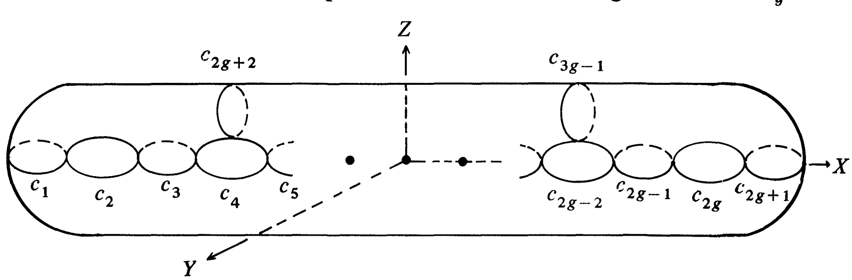

FIGURE 1. THE HANDLEBODY X_g

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second handlebody, which is so related to X_g that a translation τ parallel to the x-axis maps X_g onto X_g'. Let Φ be a homeomorphism of $\partial X_g \to \partial X_g$. Let $M = X_g \cup_\Phi X_g'$ be the 3-manifold which is obtained by identifying the boundaries of X_g and X_g' according to the rule $\tau\Phi(z) = z$, $z \in \partial X_g$. Every closed 3-manifold M admits such a representation.

Let c be a simple closed curve on ∂X_g, and let γ_c be a twist about c (see [3], [4]). It was proved in [4] that if $g > 0$, then every homeomorphism of $\partial X_g \to \partial X_g$ is isotopic to a product of twists γ_{c_i} about the curves c_i, $1 \leqq i \leqq 3g - 1$, in Figure 1.[3] We will make the assumption that our homeomorphism Φ is given as a product of the particular twists $\gamma_{c_1}, \ldots, \gamma_{c_{2g+1}}$. This involves no loss in generality if $g \leqq 2$, but if $g > 2$ the class of maps Φ which can be so represented is somewhat restricted. We are now ready to state the algorithm for deciding whether $M = X_g \cup_\Phi X_g'$ is homeomorphic to S^3.

Step 1. Given the homeomorphism

(1) $$\Phi = \gamma_{c_{\mu_1}}^{\varepsilon_1} \cdots \gamma_{c_{\mu_r}}^{\varepsilon_r}$$

where each $\varepsilon_i = \pm 1$, and each μ_i is between 1 and $2g + 1$, construct a diagram of the $(2g + 2)$-string braid

(2) $$\beta = \sigma_{c_{\mu_1}}^{\varepsilon_1} \cdots \sigma_{c_{\mu_r}}^{\varepsilon_r}$$

where σ_i is a standard generator of the braid group (see [1]). The braid σ_i is illustrated in Figure 2.

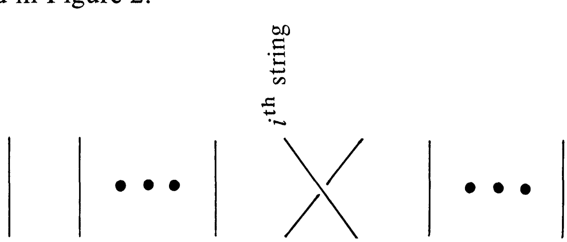

FIGURE 2. THE BRAID σ_i

Step 2. Using the braid β, construct a link L, given in projection, by joining the ends of the braid β in pairs according to the rule illustrated in Figure 3. The top of string $2i + 1$ is connected to the top of string $2i + 2$, for $i = 0, \ldots, g$; the bottom of string $2i + 1$ is connected to the bottom of string $2i + 2$ for each $i = 0, \ldots, g$. The resulting link is said to be displayed as a "plat".

[3] If $g = 0$ every homeomorphism Φ is isotopic to the identity map, the set $\{c_i\}$ is empty, and $M \sim S^3$. If $g = 1$, the twist maps γ_{c_1} and γ_{c_3} are isotopic, hence only two twist maps γ_{c_1} and γ_{c_2} are needed.

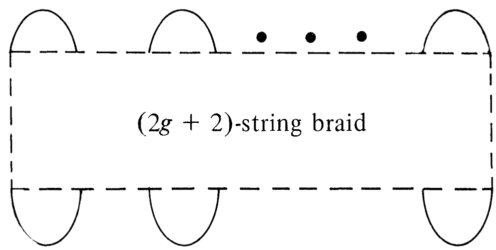

FIGURE 3. $(2g + 2)$-STRING PLAT

Step 3. Verify (by checking the projection) whether the plat L has multiplicity 1. This is a necessary condition for $M \sim S^3$. If so, apply the algorithm given by Haken in [2], or by Schubert in [5], to decide whether L is the trivial knot. We assert that $M \sim S^3$ if and only if L is the trivial knot.

3. **Sketch of proof.** We can assume without loss in generality that the embedding of X_g and X_g' in 3-space E^3 is chosen in such a way that both X_g and X_g' are invariant under a rotation Ω of 180° about the x-axis. There is also no loss in generality in assuming that the twist maps $\gamma_{c_1}, \ldots, \gamma_{c_{2g+1}}$ are defined in such a way that each γ_{c_i} commutes with the rotation Ω. Since the translation τ likewise commutes with Ω, it follows that

$$(\tau\Phi)\Omega = \Omega(\tau\Phi). \tag{3}$$

Let M/Ω be the orbit space of $M = X_g \cup_\Phi X_g'$ under the action of Ω, and let ρ be the natural projection from M to M/Ω. The condition (3) ensures that M/Ω is well defined. The quotient spaces X_g/Ω and X_g'/Ω are each homeomorphic to 3-balls, hence

$$M/\Omega = (X_g/\Omega) \bigcup\nolimits_{\rho\Phi\rho^{-1}} (X_g'/\Omega) \tag{4}$$

is represented as a genus zero Heegaard splitting, hence M/Ω must be homeomorphic to S^3. Thus the triplet $(\rho, M, M/\Omega)$ exhibits M as a 2-sheeted branched covering of S^3. The branching set is the image under ρ of the fixed point set of Ω, that is of the set $(X_g \cap x\text{-axis}) \cup (X_g' \cap x\text{-axis})$.

To understand the structure of the branching set, observe that the surface homeomorphism $\rho\Phi\rho^{-1}$ which defines the Heegaard splitting of M/Ω is a homeomorphism of $S^2 \to S^2$, and hence it is isotopic to the identity. This isotopy can be used to define a homeomorphism F of $M/\Omega \to M/\Omega$, and it can be shown that the image of the fixed point set of Ω under the product $F\rho$ is precisely the link L described in Steps 1 and 2 of the algorithm.

Suppose that M is homeomorphic to S^3. Then by a theorem of Waldhausen [7] the fixed point set of Ω must be the trivial knot, hence its image under $F\rho$ must also be trivial. Therefore a necessary condition for $M \sim S^3$ is that L have a single, unknotted component. The algorithm given in [2] and [5] enables us to test whether L is, in fact, trivial. If it is trivial, then M is the 2-fold branched covering of S^3 branched over the trivial knot. But then, $M \sim S^3$, hence the condition is also sufficient.

We remark that if Waldhausen's result [7] could be extended to transformations of period $p > 2$, then our algorithm could be extended to the class of all 3-manifolds which admit representations as p-fold branched cyclic coverings of S^3. It is not known whether this includes *all* closed 3-manifolds.[4]

4. **Related results.** The Heegaard genus of a 3-manifold M is the smallest integer g such that M admits a Heegaard decomposition $X_g \cup_\Phi X_g'$. The bridge number b of a link L is the smallest integer n such that L can be exhibited in a b-bridge presentation [6]. The braid number n of a link L is the smallest integer n such that L can be represented as a closed braid with n-strings [1]. (This is *not* the same as a "plat".)

COROLLARY 1. *Every 3-manifold of Heegaard genus $g \leqq 2$ can be exhibited as a 2-fold branched cyclic covering of S^3, branched over a knot or link of bridge number $g + 1$. The two-fold branched cyclic covering of S^3 branched over a knot or link of bridge number b is a 3-manifold of Heegaard genus $\leqq b - 1$.* (This generalizes a result due to Schubert [6].)

THEOREM 2. *The p-fold branched cyclic covering of S^3, branched over a knot of braid number n, is a 3-manifold of Heegaard genus $\leqq (p - 1)(n - 1)$, for every $p \geqq 2$.*

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[4] A new result of J. Montisinos establishes that this does *not* include all closed 3-manifolds. See J. Montisinos, *3-Variétés qui ne sont pas revêtements cycliques ramifés sur S^3*, (to appear).

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