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AUTOMORPHISM GROUPS OF PARTIAL ORDERS

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1. The automorphism group $\Gamma(P)$ of a partial order P is the collection of all order preserving permutations (automorphisms) of P, a subgroup of the symmetric group on P. If P and Q are partial orders then $P \times Q$ becomes a partial order by reverse lexicography: $(p, q) < (p', q')$ if $q < q'$ or $q = q'$ and $p < p'$. If f is a function whose domain contains the element a, we use af to denote the image of a under f.

THEOREM 1. *$\Gamma(P \times Q)$ contains an isomorphic copy of $\Gamma(P)$ wr $\Gamma(Q)$, a nonstandard wreath product of $\Gamma(P)$ by $\Gamma(Q)$.*

PROOF. Let (b, f) be an element of $\Gamma(P)$ wr $\Gamma(Q)$, with $f: Q \to \Gamma(P)$ and $b \in \Gamma(Q)$, where $\Gamma(Q)$ may be identified with a subgroup of $\mathrm{Aut}(\prod_{q \in Q} \Gamma(P))$. The action of b on f is defined by $q(fb) = (qb^{-1})f$. Define

$$\phi : \Gamma(P) \text{ wr } \Gamma(Q) \to \Gamma(P \times Q)$$

by

$$(p, q)[(b, f)\phi] = (p[qbf], qb).$$

It is not difficult to show that ϕ is an embedding.

DEFINITION. $\Gamma(P \times Q)$ is β-imprimitive if the sets $P \times \{q\}$, $q \in Q$, are sets of imprimitivity (i.e. if for all $\alpha \in \Gamma(P \times Q)$, $(p_1, q)\alpha = (p'_1, q')$ and $(p_2, q)\alpha = (p'_2, q'')$ implies $q' = q''$).

The author wishes to thank Jack Sonn for simplifying the proof of the sufficiency of the following theorem.

THEOREM 2. *$\Gamma(P \times Q) \cong \Gamma(P)$ wr $\Gamma(Q)$ if and only if $\Gamma(P \times Q)$ is β-imprimitive.*

PROOF. If ϕ is an isomorphism then every $\alpha \in \Gamma(P \times Q)$ behaves algebraically like one $(b, f) \in \Gamma(P)$ wr $\Gamma(Q)$, and β-imprimitivity follows from the definition of (b, f). Conversely, if $\Gamma(P \times Q)$ is β-imprimitive, then each $\alpha \in \Gamma(P \times Q)$ induces an $\alpha^* \in \Gamma(Q)$ and the mapping $\alpha \to \alpha^*$ is an epimorphism. It follows that the diagram

$$\begin{array}{ccccccccc} 1 & \to & \Gamma(P) & \to & \Gamma(P \times Q) & \to & \Gamma(Q) & \to & 1 \\ & & \| & & \phi\uparrow & & \| & & \\ 1 & \to & \Gamma(P) & \to & \Gamma(P) \text{ wr } \Gamma(Q) & \to & \Gamma(Q) & \to & 1 \end{array}$$

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is commutative with exact rows and, by the five lemma, ϕ is an isomorphism.

THEOREM 3. *If P is a partial order and P° is the partial order obtained by adjoining a universal lower bound to P, then $\Gamma(P^\circ) \cong \Gamma(P)$.*

THEOREM 4. *If P is a finite partial order with universal lower bound and Q is a finite partial order, then $\Gamma(P \times Q)$ is β-imprimitive.*

The proof is by induction on $h(q)$, h being the height function on $Q: h(q) = \sup\{\text{lengths of maximal subchains from } q_0 \text{ to } q\}$, where the supremum is taken over all minimal elements q_0 in Q, for which $q_0 \leqq q$.

Theorem 4 tells us that, at least in the finite case, we can be certain that the automorphism group on $P \times Q$ is the wreath product $\Gamma(P)$ wr $\Gamma(Q)$ if only P has a universal lower bound, and if P does not, then by adjunction of a lower bound, that wreath product is obtained (Theorem 3). Details of the proofs may be found in [**1**] or [**2**].

2. Having obtained a wreath product it is natural to ask whether the standard wreath product can be obtained (as an automorphism group of a partial order). Given an arbitrary group A with a well-ordered generating system $\{a_j; j \in J\}$, Frucht [**5**] has shown there is a partial order Φ_A for which $\Gamma(\Phi_A) \cong A$. Namely $\Phi_A = A \times (2 + J)$ with

$$(a, i) < (a, j), \qquad a \in A \quad \text{and} \quad i < j < 2 + J,$$

$$(a_i a, 1) < (a, 2 + j), \qquad a \in A \quad \text{and} \quad i \leqq j < 2 + J.$$

Using Frucht orders Φ_A, Φ_B for two groups A and B, with generating systems of order types J, I, respectively, we next construct a partial order whose automorphism group is the standard wreath product $A \wr B$, and which is more economical than the Frucht order for $A \wr B$.

DEFINITION. A partial order P is called uniform if (i) every subset of P has minimal elements and (ii) if $p, p' \in P$ such that $h(p) = h(p')$, where h is the height function, then p and p' are in the same orbit of $\Gamma(P)$.

Note that the orbits of $\Gamma(P)$ are well-ordered; we denote the minimal orbit by Θ_P, or if no ambiguity can arise, by Θ. As an important example, Frucht orders are uniform.

DEFINITION. If P and Q are uniform partial orders, we define

$$P \wr Q = (P \times \Theta_Q) \cup (Q - \Theta_Q),$$

with the order in $P \times \Theta_Q$ determined by that of P, the order in $Q - \Theta_Q$ remaining unchanged, and $(p, r) < q$ if $r < q$ in Q.

While $P \wr Q$ is a uniform partial order, it is not in general true that $\Gamma(P \wr Q)$ is a wreath product. However, if P is a Frucht order, we do obtain a wreath product, and if Q, too, is a Frucht order, a standard wreath product is obtained. We prove the latter first.

THEOREM 5. $\Gamma(\Phi_A \wr \Phi_B) \cong A \wr B$.

PROOF. Major steps in the proof are listed here. We use λ generically to denote an element of $W = \Gamma(\Phi_A \wr \Phi_B)$. Note that a typical element of $\Phi_A \wr \Phi_B$ has the form $((a, j), (b, 1))$, where $j < J$, $a \in A$, $b \in B$, if it belongs to $\Phi_A \times \Theta_B$ and the form (b, i), where $b \in B$ and $1 < i < I$ if it belongs to $\Phi_B - \Theta_B$, and where $\Theta_B = B \times \{1\}$ is the minimal orbit of Φ_B.

(i) If $((e, 1), (b, 1))\lambda = ((a, 1), (b', 1))$, where $e = e_A$ is the identity of A, a is in A, b and b' in B, then $((g, j), (b, 1))\lambda = ((ga, j), (b', 1))$ for all $j < 2 + J$ and all g in A, and $(b, i)\lambda = (b', i)$ for all i with $1 < i < 2 + I$.

(ii) $K = \{k \in W; (k \mid \Phi_A \times \Theta_B)\pi_1 = \pi_1\}$, where π_1 is the projection on the first component, is a subgroup of W isomorphic to B.

(iii) $F = \{f \in W; (f \mid \Phi_A \times \Theta_B)\pi_2 = \pi_2\}$ is a normal subgroup of W isomorphic to $\prod_{b \in B} A_b$, where $A_b \cong A$ for each b in B.

(iv) W is a semidirect product of F by K, and so is (isomorphic to) a semidirect product of $\prod_b A$ and B.

(v) The mapping $\alpha: K \to \mathrm{Aut}(F)$ defined by $f(k\alpha) = k^{-1}fk$ is an embedding.

Hence $W = \Gamma(\Phi_A \wr \Phi_B)$ is the relative holomorph of $\prod_b A$ by B, i.e. the standard wreath product of A by B. We point out that the groups $\Gamma(\Phi_A \wr \Phi_B)$ and $A \wr B$ are not isomorphic as permutation groups, but only as groups.

COROLLARY. *For $\lambda \in \Gamma(\Phi_A \wr \Phi_B)$, λ induces a map $\lambda^* \in \Gamma(\Phi_B)$ such that $\lambda^* \mid \Phi_B - \Theta_B = \lambda \mid \Phi_B - \Theta_B$. The mapping $\lambda \to \lambda^*$ is an epimorphism.*

The proof of the next theorem mimics that of the preceding one. Here only one of the uniform partial orders is a Frucht representation.

THEOREM 6. *Let Q be a uniform partial order. Let A be a group with Frucht representation Φ_A. Then the automorphism group of $\Phi_A \wr Q$ is a wreath product (in general, nonstandard) of A by $\Gamma(Q)$.*

Observe that $\Gamma(\Phi_A \wr Q)$ is a standard wreath product if and only if $\Gamma(Q)$ is isomorphic as a permutation group on Θ_Q to a Cayley representation. Details of the proofs of Theorems 5 and 6 may be found in [**1**] or [**3**].

3. DEFINITION. Let A be a group and P a uniform partial order with minimal orbit Θ such that $\Gamma(P) \cong A$. A group B is said to be obtainable from A if there exist a nonempty set S and a surjection $f: \Theta \to S$ such that if $Q = P \cup S$ with $p > s$ if and only if $f(p) = s$, for p in P and s in S, then $\Gamma(Q) \cong B$.

What is being done here is that a new family of minimal orbits is adjoined to a uniform partial order with each member of the original minimal orbit having a unique predecessor. (Note that the resulting order need not be uniform.) We will write the image of p in Θ under f as $f(p)$ contrary to our usual notation.

THEOREM 7. *If B is obtainable from A, then B is embedded as a subgroup of A and if $|A| \neq 2$, then every subgroup of A is obtainable from A.*

PROOF. If B is obtainable from A, then identifying A with $\Gamma(P)$ and B with $\Gamma(Q)$, the mapping $\psi : B \to A$ defined by $b\psi = b \mid P$, the restriction of b to P, is an embedding.

Conversely, let B be a subgroup of A and Φ_A the Frucht representation of A. Let $S = \{A - B\} \cup B$; i.e. the elements of S are the points of B and the set $A - B$. Define $f: \mathrm{A} \times \{1\} \to S$ by $f(\mathrm{a}, 1) = a$ if $a \in B$, and $f(a, 1) = A - B$ if $a \notin B$. If $Q = \Phi_A \cup S$ with $(a, 1) > f(a, 1)$, then $\Gamma(Q) \cong B$. In particular, $\psi : \Gamma(Q) \to A$ defined as above is an isomorphism of $\Gamma(Q)$ onto $B \subseteq A$.

4. A theorem of Kaloujnine and Krasner [**6**] states that every extension of A by B can be found embedded in $A \wr B$. Using Theorems 5 and 6 we indicate here a natural way of producing partial order representations for those extensions which are split, i.e. for the semidirect products of A by B.

Specifically, if $\Phi_A \wr \Phi_B$ is the wreath product representation of $A \wr B$ and $D = BA$ is a semidirect product of A by B with homomorphism α, we let $S = A$ and $f: \Theta \to S$ by $f((a, 1), (b, 1)) = a(b\alpha)$, where

$$\Theta = (A \times \{1\}) \times (B \times \{1\})$$

is the minimal orbit of $\Phi_A \wr \Phi_B$. We have $\alpha : B \to \mathrm{Aut}(A)$, and S and f define a new partial order Q as in the definition of "obtainable". From Theorem 6, the automorphism group of Q is (isomorphic to) a subgroup of the automorphism group of $\Phi_A \wr \Phi_B$, that is a subgroup of $A \wr B$. We show next that the new automorphism group is isomorphic to D.

THEOREM 8. *Let D be a semidirect product of A by B with homomorphism α. Let Φ_A and Φ_B be the Frucht orders of A and B respectively. Let $Q = \Phi_A \wr \Phi_B \cup A$ with $((a, 1), (b, 1)) > a(b\alpha)$. Then $\Gamma = \Gamma(Q) \cong D$.*

The proof is accomplished in a series of steps showing that (1) Γ is (isomorphic to) a subgroup of $A \wr B$; (2) for $\rho \in \Gamma$, if $((e_A, 1), (e_B, 1))\rho = ((y, 1), (z, 1))$, where e_A, e_B are the identities of A, B respectively, $y \in A$, $z \in B$, and if $a_0 = y(z\alpha)$, then $e_A\rho = y(z\alpha) = a_0$ and $a\rho = a(z\alpha)a_0$ for every a in A; (3) if $((a, 1), (b, 1))\rho = ((a', 1), (b', 1))$, where a, a' are in A and b, b' are in B, then $a' = a[a_0(b'^{-1}\alpha)]$ and $(b^{-1}b')\alpha = z\alpha$; (4) A is embedded as a normal subgroup of Γ and B as a subgroup of Γ; and finally (5) $\Gamma \cong BA = D$.

5. Having constructed a partial order Q whose automorphism group is the semidirect product of A by B, we next consider the behavior of the automorphism group of Q when we modify Q by procedures similar to those used in constructing Q itself. Details of the proofs of the following theorem and the preceding one may be found in [**1**] or [**4**].

THEOREM 9. *Let D be a semidirect product of A by B with homomorphism* α ($\alpha: B \to \mathrm{Aut}(A)$), *and let Q be the partial order representation of D constructed in Theorem* 8. *Let C be a group with Frucht representation Φ_C, $\beta: A \to \mathrm{Aut}(C)$ a homomorphism, and $P = C \cup \Phi_C \wr Q$ with $((c, 1), a) > c(a\beta)$. Then if $\Gamma = \Gamma(P)$, we have $\Gamma \cong \bar{B}AC$, where $\bar{B} = \{b \in B; (b\alpha)\beta = \beta\}$ is a subgroup of B with $(b_1a_1c_1)(b_2a_2c_2)$ given by*

$$(b_1a_1c_1)(b_2a_2c_2) = b_1b_2(a_1(b_2\alpha))a_2(c_1(a_2\beta))c_2.$$

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REFERENCES

1. E. H. Bird, *Automorphism groups of partial orders*, Ph.D. Dissertation, Adelphi University, Garden City, N.Y., 1972.

2. ———, *The automorphism group of a product of partial orders*, Discrete Math. (to appear).

3. ———, *A partial order representation of the standard wreath product*, Discrete Math. (to appear).

4. ———, *A partial order representation of the semi-direct product*, Discrete Math. (to appear).

5. R. Frucht, *On the construction of partially ordered systems with a given group of automorphisms*, Amer. J. Math. **72** (1950), 195–199. MR **11**, 320.

6. M. Krasner and L. Kaloujnine, *Produit complet des groupes de permutations et le problème d'extension des groupes*. III, Acta Sci. Math. Szeged. **14** (1951), 69–82. MR **14,** 242.

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