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A SIMPLIFIED APPROACH TO THE COMPACTIFICATION OF MAPPINGS

BY F. A. DELAHAN AND G. E. STRECKER

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Let $f: X \to Y$ be an onto mapping (i.e., continuous function between Hausdorff spaces). If inverse images of compact sets are compact, f will be said to be *compact* and if the stronger condition that f is closed and inverse images of points are compact holds, f will be said to be *perfect*. A *compactification* (resp. *perfection*) of f is a compact (resp. perfect) mapping $f^*: X^* \to Y$ such that X is densely embedded in X^* and $f^* \mid X = f$.

Whyburn [9] introduced the notion of compactification of mappings and, by means of a "unified space" construction, showed that every mapping can be compactified. Specifically, for $f: X \to Y$, the unified space Z is the disjoint union of the underlying sets of X and Y such that $Q \subset Z$ is open provided that

(i) $Q \cap X$ and $Q \cap Y$ are open in X and Y respectively, and

(ii) for any compact set $K \subset Q \cap Y$, $f^{-1}[K] \cap (X - Q)$ is compact.

Then $r: Z \to Y$ defined by

$$r(x) = f(x) \quad \text{for } x \in X, \qquad r(x) = x \quad \text{for } x \in Y$$

is continuous and $r \mid \overline{X}: \overline{X} \to Y$ is a compactification of f. (Here $\overline{X}$ denotes the closure of X in Z.) In [12] and [13] Whyburn further investigated this construction and demonstrated its usefulness.

In 1969 Cain [3] presented a general construction that assigns to any mapping $f: X \to Y$, with Y regular, and any compactification $\tilde{X}$ of X, a compactification (in fact, a perfection) $f^*: X^* \to Y$ of f. Cain has produced two different constructions of his mapping compactification. The first [3] uses extensively the idea of filter space as developed by F. J. Wagner [8], while the second [4] uses rings of continuous functions and the assumption that Y is completely regular. Both constructions are considerably more complicated than the one to be presented below. Cain [4] has characterized his perfection (in the completely regular case) as the unique one with the property that there is a mapping h of X^* into $\tilde{X}$ which leaves both the points of X fixed and is such that for each $y \in Y$, $h \mid f^{*-1}(y)$ is a homeomorphism onto the subspace of $\tilde{X}$ consisting of all accumulation points of the inverse image of the neighborhood filter of y. He studies this

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process further in [5] and shows that, if X is locally compact, his compactification of f with respect to the one-point compactification of X coincides with Whyburn's compactification of f. In particular, Whyburn's compactification is, in this case, a perfection. Dickman [6] has studied Whyburn's construction in the case where both X and Y are locally compact. Both Bauer [1] and Dickman [7] have introduced mapping compactifications similar to Whyburn's.

New approach. Let $f: X \to Y$ be a mapping and let $\tilde{X}$ be a compactification of X. Consider the graph of f, $G(f)$, as a subspace of $\tilde{X} \times Y$ and let $\hat{f}: \overline{G(f)} \to Y$ be the restriction of the projection $\pi_Y: X \times Y \to Y$ to the closure of the graph. (Note that Y need not satisfy a regularity condition.)

THEOREM. *When Y is regular, $\hat{f}$ is equivalent to Cain's perfection of f.*

This external approach to mapping compactification has several advantages:

1. It affords an analogue of Whyburn's unified space; namely, the subspace $G(f) \cup (\{p\} \times Y)$ of $\tilde{X} \times Y$, where p is the adjoined point in the one-point compactification $\tilde{X}$ of X. (This is equivalent to Whyburn's unified space when Y is locally compact.)
2. It suggests a simpler characterization; namely, $\hat{f}$ is the unique perfection with the property that there is a mapping of the domain of $\hat{f}$ into $\tilde{X}$ which both leaves the points of X fixed and is such that, for each $y \in Y$, its restriction to $\hat{f}^{-1}(y)$ is one-to-one.
3. It yields much easier proofs (usually as corollaries of well-known theorems) of most of the published properties of these compactifications. For example, proofs for both of the main theorems in [5] become almost immediate. As an illustration, we prove the major result from [5].

THEOREM (CAIN). *If $\tilde{X}$ is a metrizable compactification of X and $f^*: X^* \to Y$ is the compactification of $f: X \to Y$ determined by $\tilde{X}$, then X^* is metrizable if and only if Y is.*

PROOF. If X^* is metrizable, then so is Y since metrizability is preserved by perfect mappings. If Y is metrizable, then so is X^* $(= \overline{G(f)})$ since metrizability is finitely productive and (closed) hereditary.

4. The approach is readily applicable to more general situations, both within and outside of the realm of topology.

Details, as well as more extensive applications and some generalizations, will be given in a forthcoming paper.

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Department of Mathematics, University of Pittsburgh, Pittsburgh, Pennsylvania 15213

Department of Mathematics, Kansas State University, Manhattan, Kansas 66506