BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 5, September 1973

FUNDAMENTAL GROUPS, NILMANIFOLDS AND ITERATED INTEGRALS[1]

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Communicated by S. S. Chern, January 20, 1973

Let X be a connected C^∞ manifold. Denote by $P(X)$ the total space of piecewise smooth paths in X. Choose a base point x_0. Denote by $P(X; x_0)$ (resp. ΩX) the space of piecewise smooth paths (resp. loops) from the base point x_0.

Let k be the field of real (or complex) numbers. All differential forms are k-valued. Let $w_1, w_2, \ldots$ denote 1-forms on X. For a piecewise smooth path $\alpha: I \to X$, let $f_i(t) = w_i(\alpha(t), \dot{\alpha}(t))$ be the value of the 1-form w_i at the tangent vector $\dot{\alpha}(t)$ of X. Define the r-time iterated integral $\int w_1 \cdots w_r$ to be the k-valued function on $P(X)$ whose value at α is given by

$$\left\langle \int w_1 \cdots w_r, \alpha \right\rangle = \int_0^1 \int_0^{t_r} \cdots \int_0^{t_2} f_1(t_1)\, dt_1 \cdots f_{r-1}(t_{r-1})\, dt_{r-1} f_r(t_r)\, dt_r$$

when $r > 0$ and $= 1$ when $r = 0$. At times, we shall also take $\int w_1 \cdots w_r$ as its restriction on ΩX or $P(X; x_0)$.

Let F be the function algebra on $P(X)$ consisting of those functions whose value at each path α remains invariant under any piecewise smooth homotopy of α relative to $\dot{I}$. In this note, we shall consider the subspace of F whose elements are linear combinations of iterated integrals. A characterization of this subspace in terms of the fundamental group $\pi_1(X)$ will be given.

We begin with a differential graded subalgebra A of the exterior algebra $\Lambda(X)$. The following assumptions are made:

I. $dA^0 = A^1 \cap d\Lambda^0(X)$.

II. $\dim H^1(A) < \infty$.

III. The canonical homomorphism $H^q(A) \to H^q(X; k)$ is an isomorphism when $q = 1$ and is a monomorphism when $q = 2$.

A primary example is the case of $A = \Lambda(X)$.

For $s \geqq 0$, denote by $F_A(s)$ the subspace of F whose elements are linear combinations of iterated integrals of the type

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AMS (MOS) subject classifications (1970). Primary 58A99; Secondary 49F05, 53C65.

Key words and phrases. Iterated integrals, path spaces, fundamental groups, differential forms, nilpotent Lie groups, nilmanifolds.

[1] Work supported in part by the National Science Foundation under NSF-GP-34257.

$$\int w_1 \cdots w_r, \qquad 0 \leqq r \leqq s, \qquad w_1, \ldots, w_r \in A^1.$$

Then $k = F_A(0) \subset \cdots \subset F_A(s) \subset \cdots$. Moreover $F_A = \bigcup F_A(s)$ turns out to be closed under multiplication. Let $F'_A(s)$ (resp. $F''_A(s)$) be obtained from $F_A(s)$ by restricting to ΩX (resp. $P(X; x_0)$). Then $F'_A = \bigcup F'_A(s)$ and $F''_A = \bigcup F''_A(s)$ are algebras obtained from the algebra F_A by restrictions.

Let $k\pi_1(X)$ be the group algebra of $\pi_1(X)$ over k, and let N be the augmentation ideal, which is generated by all $\langle\alpha\rangle - 1$, where $\langle\alpha\rangle$ denotes the homotopy class of a piecewise smooth loop at x_0. There is a pairing $F'_A \times k\pi_1(X) \to k$ given by $(u, \langle\alpha\rangle) \mapsto \langle u, \alpha\rangle$ which is the value of the linear combination u of iterated integrals at the loop α.

THEOREM 1. *With respect to the above pairing* $F'_A(s) = (N^{s+1})^{\perp}$, $s \geqq 0$.

In order to outline a proof of this theorem, we recall that iterated integrals can be defined for forms of higher degrees in A. In relation to ΩX, such iterated integrals form a differential graded algebra A' with an ascending filtration $\{A'(s)\}$ (see [3]). Choose a suitable cubical chain complex $C_*(\Omega X)$ so that it has a descending filtration by the powers of its augmentation ideal. Let $\{B(s)\}$ be the dual ascending filtration for the cochain complex $C^*(\Omega X; k)$. Then $B = \bigcup B(s)$ is a filtered subcomplex of $C^*(\Omega X; k)$. Theorem 1 follows from comparing the spectral sequences of the filtered cochain complexes A' and B and the fact that $F'_A \approx H^0(A')$. Since the restriction map $F''_A \to F'_A$ is surjective, we are also led to the next conclusion.

THEOREM 2. *If* A^0 *separates points of* X *and if* $\bigcap N^s = 0$, *then* F''_A, *taken as an algebra of functions on the universal covering space* $\tilde{X}$ *of* X *separates points of* $\tilde{X}$.

This result is related to a work of Parsin [5]. He considered the case where X is a Riemann surface, and A is the algebra of holomorphic differential forms. Our assumption III does not hold for his case.

If $\pi_1(X)$ is finitely generated torsion free nilpotent, we can show that F'_A is the coordinate ring of a simply connected nilpotent Lie group G having a uniform discrete subgroup $\Gamma \approx \pi_1(X)$. By constructing an injection $F'_A \to F''_A$, we obtain the next assertion.

THEOREM 3. *If* X *is a connected* C^∞ *manifold with* $\pi_1(X)$ *being finitely generated torsion free nilpotent, then there exists a compact nilmanifold* $M(X)$ *and a* C^∞ *map* $X \to M(X)$ *which induces an isomorphism for the fundamental groups.*

Observe that G can be taken as the Malcev completion of $\Gamma \approx \pi_1(X)$.

The nilmanifold $M(X) = G/\Gamma$ associated to X is essentially unique (see [**1**] or [**6**]).

REMARK. In the case of X being Riemannian, there is a canonical injection $F'_A \to F''_A$ with $A = \Lambda(X)$ so that, in the theorem, the map $X \to M(X)$ is canonical. If X is, moreover, real analytic, so is the map.

The details of these theorems in a somewhat more general context will appear elsewhere.

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