BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 5, September 1973

AN ILL POSED PROBLEM FOR A HYPERBOLIC EQUATION NEAR A CORNER[1]

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Communicated by Eugene Isaacson, January 23, 1973

The purpose of this note is to give a simple example of an ill posed problem for a hyperbolic equation to be solved in a region whose boundary has a corner. In [2] we gave necessary and sufficient conditions for existence, uniqueness, and the validity of certain energy estimates for the solutions of a general class of these problems. Analogous conditions for problems in regions with smooth boundaries were obtained by Kreiss [1]. Our example below is somewhat unusual in that bounded C^∞ initial data lead to a solution which is exponentially unbounded at the corner for any positive time.

Consider the equation

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_y \tag{1}$$

to be solved for the complex valued functions u and v in the region $0 < x, y, t$ with initial conditions

$$u(x, y, 0) = \Phi(x, y), \qquad v(x, y, 0) = \psi(x, y), \tag{2}$$

and boundary conditions

$$\text{(a)} \quad u(0, y, t) = av(0, y, t), \qquad \text{(b)} \quad v(x, 0, t) = bu(x, 0, t). \tag{3}$$

a and b are complex numbers.

We have the following:

THEOREM. *The above problem is well posed, i.e. generates a strongly continuous semigroup for $t > 0$ on L_2, if and only if $|ab| \leqq 1$.*

We note here that by the results in [1], the half space problem (1), (2) to be solved for $0 < x, t$; $-\infty < y < \infty$ is well posed for any boundary condition (3)(a), as is the half space problem for $0 < y, t$; $-\infty < x < \infty$ for any boundary condition (3)(b).

AMS (MOS) subject classifications (1970). Primary 35L50, 35L30; Secondary 78A45.

Key words and phrases. Hyperbolic equations, initial boundary conditions, well posedness, energy estimates.

[1] Research supported under N.S.F. Grant No. GP29-273.

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PROOF. If $|ab| > 1$, consider the functions

$$u(x, y, t) = a(ab)^{(t-x)/[2(x+y)]}, \qquad v(x, y, t) = (ab)^{(t+x)/[2(x+y)]}, \tag{4}$$

where the same argument for ab is chosen in both expressions. This pair of functions satisfies the conditions of (1) and (3) with initial data which are bounded and C^∞ for $x, y > 0$. The initial data are then multiplied by the factor $(ab)^{t/[2(x+y)]}$ which is exponentially unbounded, as is the solution, when $x + y \to 0$ for any positive t. The solution is well behaved for finite t away from $x = y = 0$.

If $|ab| \leqq 1$, we choose two positive numbers c_1, c_2 such that $c_1 |a|^2 \leqq c_2$ and $c_2 |b|^2 \leqq c_1$. We then have, using (1) and (3):

$$\frac{d}{dt} \int_0^\infty \int_0^\infty [c_1 |u|^2 + c_2 |v|^2] \, dx \, dy \leqq 0. \tag{5}$$

L_2 well posedness is immediate.

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