Let $S^1$ denote the multiplicative group of complex numbers of norm 1. Let $X$ denote a smooth $S^1$ manifold, i.e., $X$ consists of an underlying smooth manifold denoted by $|X|$ together with a smooth action of $S^1$. The equivariant complex $K$ theory of $X$ is $K^*_S(X) = K^0_S(X) \oplus K^1_S(X)$. It is a module over $R(S^1)$ the complex representation ring of $S^1$. This is the ring $\mathbb{Z}[t, t^{-1}]$. For our purposes there are two important sets of prime ideals in $\mathbb{Z}[t, t^{-1}]$:

(i) the set $P_1$ consisting of the principal ideals of the form $p = (\Phi_p(t))$ generated by the cyclotomic polynomial $\Phi_p(t)$ associated to the prime $p$ i.e., $P_1 = \{(\Phi_p(t)) \mid \forall$ primes $p$ and integers $r\}$.

(ii) the set $P = \{(\Phi_m(t)) \mid \forall$ positive integers $m\}$.

The localized ring $R(S^1)_p$ is denoted by $R$. It is the subring of the field of fractions of $R(S^1)$ consisting of fractions $a/b$ with $b$ prime to all the ideals of $P$. Let $K^*_S(X)_p = K^*_S(X) \otimes_{R(S^1)} R$. The Atiyah-Singer index homomorphism $[1]$ $Id^*_S: K^*_S(TX) \to R(S^1)$ induces a homomorphism

$$Id^X: K^*_S(TX)_p \to R.$$ 

Here $TX$ is the tangent bundle of $X$ and $|X|$ is compact without boundary. Suppose that $|X|$ is a spin$^c$ manifold. Then there is an isomorphism

$$K^*_S(X)_p \xrightarrow{\Delta^X} K^*_S(TX)_p$$

of $R$ modules [6] and we can define an $R$ valued bilinear form $\left< \cdot, \cdot \right>_X$ on $K^*_S(X)_p$ by

$$\left< a, b \right>_X = Id^X(\Delta^X(a) \cdot b).$$

**Theorem 1** [2]. The bilinear form $\left< \cdot, \cdot \right>_X$ is nonsingular, i.e., the associated homomorphism

$$K^*_S(X)_p \xrightarrow{\Phi^X} \text{Hom}_R(K^*_S(X)_p, R)$$

is surjective where $\Phi^X(a)[b] = \left< a, b \right>_X$.

This result was conjectured in a similar form in [6].

A useful consequence of Theorem 1 is this: Set $K^*_S(T_X) = K^*_S(X)_p/T_X$ where $T_X$ denotes the $R$ torsion subgroup of $K^*_S(X)_p$. The bilinear form
\( < >_X \) defines a bilinear form again denoted by \( < >_X \) on \( \tilde{K}^*_S(I)(X) \).

**Theorem 1'.** The associated homomorphism \( \tilde{K}^*_S(I)(X) \rightarrow \text{Hom}_R(\tilde{K}^*_S(I)(X), R) \) is an isomorphism.

Before mentioning applications, let me discuss the problems to which we wish to apply this result.

(i) If \( S^1 \) acts effectively on \( M \) and if \( N \) is homotopy equivalent to \( M \), does \( S^1 \) act effectively on \( N \)?

(ii) If \( S^1 \) acts effectively on a smooth manifold, what are the relations among the representations of \( S^1 \) on the tangent spaces at the points fixed by \( S^1 \) and the global invariants of the manifold, e.g., its Pontryagin classes and its cohomology?

Towards answering these questions, we introduce the set \( S_S(Y) \) associated to the closed \( S^1 \) manifold \( Y \). It consists of equivalence classes of pairs \( (X,f) \) where \( f: X \rightarrow Y \) is an equivariant map such that

1. \( |f|: |X| \rightarrow |Y| \) is a homotopy equivalence;
2. \( |f^S|: |X^S| \rightarrow |Y^S| \) is a homotopy equivalence.

Two pairs \( (X_i,f_i), i = 0, 1, \) are equivalent if there is an \( S^1 \) homotopy equivalence \( \phi: X_0 \rightarrow X_1 \) such that \( f_1 \phi = S^1 \) homotopic to \( f_0 \). The equivalence class of \( (X,f) \) is denoted by \( [X,f] \).

Suppose that \( |Y| \) is a spin\(^c\) manifold. Then if \( [X,f] \in S_S(Y), |X| \) is a spin\(^c\) manifold and we can define an induction homomorphism \([3]\)\[
\langle f^*(1_x), y \rangle_Y = \langle x, f^*(y) \rangle_X .
\]
If \( 1_X \) denotes the identity of the algebra \( \tilde{K}^*_S(I)(X) \), then the element \( f^*(1_X) \) is a very important geometric invariant of the situation. It relates the algebra \( \tilde{K}^*_S(I)(Y) \) with the differential structures on \( |X| \) and \( |Y| \) and with the representations of \( S^1 \) on the normal bundles to the fixed sets \( X^S \subset X \) and \( Y^S \subset Y \). In order to illustrate these relations in a simple manner, we restrict ourselves to the case where \( Y^S \) consists of isolated points. In addition, we want to assume that the odd dimensional rational cohomology of \( |Y| \) vanishes. In this situation the natural homomorphism \( K^*_S(Y) \rightarrow K^*(|Y|) \) induces a homomorphism \( \tilde{K}^*_S(Y) \rightarrow K^*(|Y|) \otimes Q \) and the composition with the Chern character isomorphism \( ch \) to \( H^*(|Y|, Q) \) is denoted by \( \phi_Y \). If \( p \in Y^S \), the representation of \( S^1 \) on the normal bundle of \( Y^S \) at \( p \) is denoted by \( NY_p \). We may assume it to be a complex representation of \( S^1 \).

We remark that if \( [X,f] \in S_S(Y), f^S: X^S \rightarrow Y^S \) is a homeomorphism when \( Y^S \) consists of isolated points. Let \( g: |Y| \rightarrow |X| \) be a homotopy inverse to \( |f| \). We can now illustrate the geometric importance of \( f^*(1_X) \) and its relation with the algebra \( \tilde{K}^*_S(Y) \).
THEOREM 2 [5]. Let \([X, f] \in S_{S^1}(Y)\). Then \(\phi_*(f_*(1)) = g^*A(|X|/A(Y))\) where \(A(|X|)\) denotes the cohomology class associated to the tangent bundle of \(|X|\) by the power series \((x/2)/\sinh x/2\).

THEOREM 3 [5]. Let \(q \in X^{S^1}\). The restriction of \(f_*(1)\) to \(p \in Y^{S^1}\) is denoted by

\[f_*(1)_p \in K_{S^1}^*(p) = R \quad \text{and} \quad f_*(1)_f(q) = \pm t^{N_q} \cdot \lambda^{-1}(NY_{f(q)})/\lambda^{-1}(NX_q) \in R.\]

Here \(N_q\) is an integer and \(\lambda^{-1}(NX_q) = \sum (-1)^i \lambda^i(NX_q) \in R.\)

THEOREM 4 [5]. \(f_*(1)_f(q)\) is a unit of \(R_{P^1}\). (Compare [6, p. 139, Theorem 2.6].)

THEOREM 5 [5]. If \(f^* : \tilde{K}_{S^1}^*(Y) \to \tilde{K}_{S^1}^*(X)\) is an isomorphism, \(f_*(1)\) is a unit of \(\tilde{K}_{S^1}^*(Y)\) and \(f_*(1)_q = \pm 1 \in R\) for all \(q \in X^{S^1}\) and \(\phi_Y f_*(1) = 1 \in H^*(|Y|, \mathbb{Q}).\)

Briefly, Theorem 2 relates \(f_*(1)\) and Pontryagin classes, Theorem 3 relates \(f_*(1)\) and normal representations and Theorems 4 and 5 relate \(f_*(1)\) with the algebra \(K_{S^1}^*(Y)\). We remark that Theorem 5 together with Theorem 2 actually implies that if \(f\) is an \(S^1\) homotopy equivalence, \(|f|\) preserves Pontryagin classes.

Here is an interesting example to illustrate the ideas. Let \(p, q\) be relatively prime integers. Choose integers \(a, b\) such that \(-ap + bq = 1.\) Let \(N = t^p + t^q\) and \(M = t^1 + t^pq\) denote the indicated complex 2 dimensional representations of \(S^1\). The one point compactifications \(N^+\) and \(M^+\) are smooth \(S^1\) manifolds with \(|N^+| = |M^+| = S^4.\) The map \(\Phi : N \to M\) defined by \(\Phi(z_0, z_1) = (z_0^a z_1^b, z_0^a + z_1^b)\) is proper, hence defines a map \(\Phi^* : N^+ \to M^+\) and \([N^+, \Phi^+] \in S_{S^1}(M^+).\) The invariant \((\Phi^*)_*(1_{N^+})\) is

\[(1 - t)(1 - t^{pq})/(1 - t^p)(1 - t^q) \cdot 1_{M^+} \in K_{S^1}^*(M^+).\]

For deeper applications of ideas, see [4] and [5].

Theorems 3 and 4 combine to give a comparison of the representations \(NX_q\) and \(NY_{f(q)}\) as follows: Let \(F\) denote the field of fractions of \(\mathbb{Z}[t, t^{-1}].\) For each prime ideal \(p\) of \(P\) let \(\|\|_p\) denote the valuation defined by \(p.\) We interpret this as a norm on \(F.\) Then

THEOREM 6. If \([X, f] \in S_{S^1}(Y)\) and \(q \in X^{S^1}\),

\[\|\lambda^{-1}(NX_{f(q)})/\lambda^{-1}(NX_q)\|_p = 1 \quad \text{for all} \quad p \in P_1.\]

REMARK. If this is true for all \(p \in P,\) then the real representations of \(NY_{f(q)}\) and \(NX_q\) are equal.
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