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ON FOURIER COEFFICIENTS OF EISENSTEIN SERIES[1]

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In this paper we wish to prove that under certain conditions the Fourier coefficients of the Eisenstein series for an arithmetic group acting on a tube domain are all rational numbers.

Let G be a connected, simply-connected, semisimple, and almost $\boldsymbol{Q}$-simple linear algebraic group defined over the rational number field $\boldsymbol{Q}$. Let $\boldsymbol{R}$ be the real number field. Then $G_{\boldsymbol{R}}$ is connected, and we assume that if K is a maximal compact subgroup of it, then $\mathfrak{X} = K \backslash G_{\boldsymbol{R}}$ is a non-compact hermitian symmetric space. We also assume that the $\boldsymbol{Q}$-rank of G is positive and that the $\boldsymbol{Q}$-root system ${}_{\boldsymbol{Q}}\Sigma$ of G is of type C. Then the relative $\boldsymbol{R}$-root system ${}_{\boldsymbol{R}}\Sigma$ of G is of type C and therefore $\mathfrak{X}$ is isomorphic to a tube domain

$$(1) \qquad \mathfrak{T} = \{X + iY \in \boldsymbol{C}^m \mid Y \in \mathfrak{R}\},$$

where $\boldsymbol{C}$ is the complex number field, and $\mathfrak{R}$ is a homogeneous, self-adjoint cone in $\boldsymbol{R}^m$. Moreover, it follows that an isomorphism of $\mathfrak{X}$ with $\mathfrak{T}$ may be chosen such that

$$(2) \qquad P_{\boldsymbol{R}} = \{g \in G_{\boldsymbol{R}} \mid g \text{ acts by linear affine transformation on } \mathfrak{T}\},$$

then $P_{\boldsymbol{R}}$ is a $\boldsymbol{Q}$-parabolic subgroup of $G_{\boldsymbol{R}}$, and every element of the unipotent radical $U_{\boldsymbol{R}}$ of $P_{\boldsymbol{R}}$ acts by a real translation. Finally, we may write $G = R_{K/Q}\, G^*$, where G^* is absolutely almost simple, and defined over a totally real algebraic number field $\boldsymbol{K}$, and $R_{K/Q}$ is the ground field reduction functor [9, Chapter 1].

Let $\Gamma \subset G_{\boldsymbol{R}}$ be an arithmetic subgroup of G_C, then $\Lambda' = \Gamma \cap U$ is a lattice in $U_{\boldsymbol{R}}$. If F is an automorphic form of even weight l with respect to Γ, i.e.,

$$(3) \qquad F(Z \cdot \gamma) j(Z, \gamma)^l = F(Z), \quad \text{for all } \gamma \in \Gamma,$$

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[1] The author wishes to thank his thesis advisor, Professor W. L. Baily, Jr., for his assistance throughout this work.

where $j(Z, g)$ is the functional determinant of $g \in G_{\boldsymbol{R}}$ at $Z \in \mathfrak{T}$, then

(4) $$F(Z + S) = F(Z), \quad \text{for all } S \in \Lambda'.$$

Therefore we have the Fourier expansion:

(5) $$F(Z) = \sum_{T \in \Lambda} a(T)e^{2\pi i(T,Z)},$$

where Λ is the dual lattice of Λ' with respect to a nondegenerate symmetric bilinear form (,) on U, with respect to which $\mathfrak{R}$ is selfadjoint.

If $a \in G_Q$, define ${}^aP = aPa^{-1}$, and $\Gamma_{0,a} = \Gamma \cap {}^aP$, $\Gamma_0 = \Gamma_{0,e} = \Gamma \cap P$. Since P is a $\boldsymbol{Q}$-parabolic subgroup, we have $G_Q = \bigcup_{a \in A} \Gamma a P_Q$, $A \subset G_Q$ being a finite set.

For each $a \in G_Q$, we may form an Eisenstein series

(6) $$E_{l,a}(Z) = \sum_{\gamma \in \Gamma/\Gamma_{0,a}} j(Z, \gamma a)^l;$$

it is an automorphic form of weight l with respect to Γ.

Now, for each $a \in G_Q$, we may choose a number $c(a)$ and form the Eisenstein series

(7) $$E_l(Z) = \sum_{a \in A} E_{l,a}(Z)c(a)^l.$$

We explain briefly how the $c(a)$ are chosen. Namely, we first consider Eisenstein series on the adele group G_A of G and then view these as automorphic forms on $\mathfrak{T}$. The result is linear combinations of the form (7), and these linear combinations are independent of the choice of the set A of representatives, whence a definition of $c(a)$ for every $a \in G_Q$.

The main object of this work is to see that if $c(a)$ are chosen in this way, then the Fourier coefficients $a_l(T)$ of the Fourier expansion of $E_l(Z)$,

(8) $$E_l(Z) = \sum_{T \in \Lambda} a_l(T)e^{2\pi i(T,Z)},$$

are all rational numbers.

Tube domains. Any tube domain is a direct product of irreducible tube domains. The irreducible tube domains may be divided into the following five types:

A. $\mathfrak{R}$ is the cone of $n \times n$ positive definite hermitian symmetric matrices ($m = m^2$).

B. $\mathfrak{R}$ is the cone of $n \times n$ positive definite real symmetric matrices ($m = n(n + 1)/2$).

C. $\mathfrak{R}$ is the cone of $n \times n$ positive definite quaternion hermitian symmetric matrices ($m = n(2n - 1)$).

D. $\mathfrak{R}$ is the cone of the points $(x_1, \ldots, x_n)$ in the n-dimensional real vector space such that $x_1^2 > \sum_{i=2}^n x_i^2, x_1 > 0$ $(m = n)$.

E. $\mathfrak{R}$ is the cone of 3×3 positive definite Cayley hermitian symmetric matrices $(m = 27)$.

Example and historical background. Let $\mathfrak{T}$ be Siegel's generalized half plane of degree n; it is the irreducible tube domain of type B for this n. We have $G_{\boldsymbol{R}} = \mathrm{Sp}(n, \boldsymbol{R})$. If $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}(n, \boldsymbol{R})$, and $Z \in \mathfrak{T}$, then

$$Z \cdot \begin{pmatrix} A & B \\ C & D \end{pmatrix} = (ZB + D)^{-1}(ZA + C). \tag{9}$$

We have

$$\begin{aligned} P_{\boldsymbol{R}} &= \left\{ \begin{pmatrix} A & 0 \\ {}^tA^{-1}S & {}^tA^{-1} \end{pmatrix} \middle| \, S\text{:symmetric} \right\}, \\ U_{\boldsymbol{R}} &= \left\{ \begin{pmatrix} 1 & 0 \\ S & 1 \end{pmatrix} \middle| \, S\text{:symmetric} \right\}. \end{aligned} \tag{10}$$

If we let $\Gamma = \mathrm{Sp}(n, \boldsymbol{Z})$, the symplectic modular group, then $\Lambda' =$ integral symmetric matrices, $\Lambda =$ half-integral symmetric matrices. The finite set A contained in G_Q in this case is $\{e\}$.

C. Siegel [8] proved the rationality of $a_l(T)$ in the above case in 1939. H. Braun [4] dealt with some special cases of the tube domains of type A in 1949, and H. Klingen [6] dealt with the Hilbert modular group (a special case of type B) in 1960, and then W. Baily, Jr. dealt with an irreducible case of type E in 1970 [3].

Many ideas of the proof, in this paper, of the same result for a more general class of tube domains and arithmetic subgroups come from the ideas of the proof in that paper by W. Baily, Jr.

We now sketch the order of developments in this paper. At first, we describe the relationships between tube domains and Jordan algebras [7], and then discuss the structures of Jordan algebras. In particular, we describe the $\boldsymbol{R}$- and $\boldsymbol{K}$-Jordan algebra structures of U as well as the notions of determinant and rank of a class of Jordan algebras. Later we need certain facts about parabolic subgroups, functional determinant and boundary components and the relationships among them, which enable us, by way of the Bruhat decomposition, to reduce our treatment of the Fourier coefficients, by induction, to those associated to the "biggest cell". That is, if i is the element of $G_{\boldsymbol{R}}$ sending each $Z \in \mathfrak{T}$ to $-Z^{-1}$, then it suffices to treat the Fourier coefficients of

$$E_l^{(n)} = \sum_{g \in P_Q i P_Q / P_Q} j(Z, g)^l c(g)^l = \sum_{T \in \Lambda} a_l^{(n)}(T) e^{2\pi i (T, Z)}. \tag{11}$$

It is proved that $j(Z, i) = |Z|^{-N}$, where $|Z|$ is the determinant of Z and $N = (n-1)n_0 + 2, n_0$ being equal to $2, 1, 4, n-2$ or 8 corresponding to types A, B, C, D or E respectively. Hence $E_l^{(n)}(Z)$ can be written as

$$E_l^{(n)}(Z) = \sum_{X \in U_Q} |Z + X)^{-Nl} c(t_X i)^l, \tag{12}$$

where t_X is the real translation by X.

Then, by applying the Gamma integral [**5**] and the Poisson summation formula (cf. [**8**, §7], [**3**, §9.3]), we have

$$\begin{aligned} a_l^{(n)}(T) = {} & v(\Lambda)^{-1}(2\pi i)^{dnNl}\pi^{-dn(N-2)/4}\prod_{j=0}^{n-1}\gamma(Nl - jn_0/2)^{-d} \\ & \cdot |T|^{Nl-N/2}\sum_{X \in U_Q/\Lambda'} e^{2\pi i(T,X)}c(t_X i)^l, \end{aligned} \tag{13}$$

where $d = [\boldsymbol{K}:\boldsymbol{Q}]$, $\gamma(\ \ )$ is the Γ-function and $v(\Lambda)$ is the volume of the fundamental set of Λ.

It remains to calculate $S = \sum_{X \in U_Q/\Lambda'} e^{2\pi i(T,X)}c(t_X i)^l$. We note that $c(t_X i)$ is the content of X in Siegel's case. Following the same idea as in ([**8**, §7], [**3**, §9.3]), we write $S = \prod_p S_p$, $S_p = \prod S_{\mathfrak{p}}$, where p runs over all finite prime numbers and $\mathfrak{p}$ over all prime ideals of $\boldsymbol{K}$ dividing p. Applying Hensel's lemma [**1**], we may prove that, for all prime ideals $\mathfrak{p}$ of $\boldsymbol{K}$, $S_{\mathfrak{p}}$ is rational, and for all but a finite number of prime ideals $\mathfrak{p}$, we have

$$S\mathfrak{p} = \sum_{X \in i_{p^\alpha}} \exp((2\pi i/p)\mathrm{tr}_{F_{p^\alpha}/F_p}(T, X))_{p^{-\alpha R}}{}_{(X)Nl}, \tag{14}$$

provided that the $\boldsymbol{K}$-structure of the Jordan algebra is of a certain form, which includes tube domains of types B, C, D and E and some cases of type A, where $\alpha = [\boldsymbol{K}\mathfrak{p}:\boldsymbol{Q}p]$, and $\mathfrak{j}_{p^\alpha}$ is the Jordan algebra over the finite field $\boldsymbol{E}_{p^\alpha}$ of p^α elements, obtained in a certain way from the $\boldsymbol{K}$-structure of the original Jordan algebra, and $R(X)$ is the rank of X.

We then proceed to the explicit calculation of exponential sums $S\mathfrak{p}$, with the help of the technical lemmas proved earlier. The final results are (for all but a finite number of prime ideals $\mathfrak{p}$):

(i) n_0 even. $S\mathfrak{p} = \prod_{k=0}^{n-1}(1 - (\sigma \boldsymbol{G}^{\alpha n_0})^k p^{-\alpha Nl})$,

(ii) type D, n_0 odd. $S\mathfrak{p} = (1 + \sigma \boldsymbol{G}^{\alpha(n_0+1)}(|T|/p)_\alpha p^{-\alpha Nl})(1 - p^{-\alpha Nl})$,

(iii) type B, $n = 2r + 1$.

$$S\mathfrak{p} = (1 - p^{-\alpha Nl})\prod_{k=1}^{r}(1 - \boldsymbol{G}^{4k\alpha}p^{-2\alpha Nl}),$$

(iv) type B, $n = 2r$.

$$S\mathfrak{p} = (1 - p^{-\alpha Nl})(1 + \boldsymbol{G}^{2\alpha n}(|T|/p)_\alpha p^{-\alpha Nl})\prod_{k=1}^{r-1}(1 - \boldsymbol{G}^{4\alpha k}p^{-2\alpha Nl}),$$

where $G = \sum_{x \in F_{p^\alpha}} \exp(2\pi i/p)x^2$ is the Gaussian sum, $(a/p)_\alpha = 1$ or -1 according as the number $a \in E_{p^\alpha}$ is a square or a nonsquare, and $\sigma = (|f|/p)_\alpha$, $|f|$ being the determinant of the quadratic form f of the entries of the Jordan algebra $\mathfrak{j}_{p^\alpha}$.

By calculating $v(\Lambda)$ and using the values of $S\mathfrak{p}$ and of L-functions we are able to prove that the Fourier coefficients in the cases we are considering are all rational numbers.

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