INDEX THEORY FOR SINGULAR QUADRATIC FUNCTIONALS IN THE CALCULUS OF VARIATIONS\footnote{The author is indebted to Professor Magnus R. Hestenes for suggesting this problem and for his suggestions in its preparation.}

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1. Introduction. Let $P$, $Q$, and $R$ be real-valued $n \times n$ matrix functions defined on the interval $[a, b]$. Assume that $P$, $Q$, and $R$ are continuous on $[a, b)$ and that $P(t)$ and $R(t)$ are symmetric matrices for each $t$ in $[a, b)$. We do not assume that $Q$ is symmetric. Also assume that $R$ has the property that its value for any $t$ in $[a, b)$ is positive definite, that is, $v^*R(t)v > 0$ for all $n$-vectors $v \neq 0$ and for each $t$ in $[a, b)$. Let

$$J(x, y) |_{e_1}^{e_2} = \int_{e_1}^{e_2} \left[ \dot{x}^*(t)R(t)\dot{y}(t) + x^*(t)Q(t)\dot{y}(t) + \dot{x}^*(t)Q^*(t)y(t) \right] dt (a \leq e_1 \leq e_2 < b),$$

(1.1)

for $x$ and $y$ in the class $A$ of vector-valued functions described below. Also let

$$J_e(x, y) = J(x, y) |_a^e,$$

$$J_e(x) = J_e(x, x),$$

(1.2)

$$J(x, y) = \liminf_{e \to b} J_e(x, y),$$

(1.3)

$$J(x) = \liminf_{e \to b} J_e(x)$$

for $x$ and $y$ in $A$. The class $A$ is the set of vector-valued functions $x^*(t) = (x_1(t), \ldots, x_n(t)), a \leq t \leq b$, satisfying

(i) $x(t)$ is continuous on the interval $[a, b]$ and $x(a) = x(b) = 0$,

(ii) $x(t)$ is absolutely continuous and $\dot{x}^*(t)x(t)$ is Lebesgue integrable on each closed subinterval of $[a, b]$. $A$ is a vector space of functions.

$J$ is said to be singular at a point $t$ in $[a, b]$ if the determinant of $R(t)$ is zero or not defined. The point $t = b$ is a singular point in this paper.

2. Preliminaries. What is presented here is part of a quadratic form theory developed and used extensively by Hestenes [3], [4]. Let $Q(x)$
be a quadratic functional defined on a vector space $V$ and let $Q(x, y)$ be its associated symmetric bilinear functional. Two vectors $x$ and $y$ in $V$ are said to be $Q$-orthogonal whenever $Q(x, y) = 0$. A vector $x$ is said to be $Q$-orthogonal to a subset $S$ of $V$ whenever $Q(x, y) = 0$ for every $y$ in $S$. By the $Q$-orthogonal complement $S^Q$ of the set $S$ in $V$ is meant the set of all vectors $x$ in $V$ that are $Q$-orthogonal to $S$. $S^Q$ is a subspace of $V$. A vector in $S$ that is $Q$-orthogonal to $S$ is called a $Q$-null vector of $S$. The intersection $S \cap S^Q$ is the set of $Q$-null vectors of $S$ and is usually denoted by $S_0$. If $S$ is a subspace of $V$, then so is $S_0$.

Let $S$ be any subspace in $V$. We define the nullity $n(S)$ of $Q$ on $S$ or more simply the $Q$-nullity of $S$ to be the dimension of the subspace $S_0 = S \cap S^Q$ of $Q$-null vectors in $S$. We define the signature $s(S)$ of $Q$ on $S$, the index of $Q$ on $S$, or the $Q$-signature of $S$ to be the dimension of a maximal subspace $M$ of $S$ on which $Q < 0$ if this dimension is finite. If no such finite dimensional space exists, we set $s(S) = \infty$. By $Q < 0$ on $M$ we mean that $Q(x) < 0$ for each nonzero $x$ in $M$. It turns out that the dimension $s(S)$ of $M$ is independent of the choice of $M$ so that the notion of signature is well defined.

**Theorem 2.1.** If the $Q$-signature of $S$ is finite where $S$ is a subspace of $V$, then it is given by one of the following quantities:

(i) the dimension of a maximal subspace $M$ in $S$ on which $Q < 0$;

(ii) the dimension of a maximal subspace $M$ of $S$ on which $Q \leq 0$ and having $M \cap S_0 = 0$;

(iii) the dimension of a minimal subspace $M$ of $S$ such that $Q \geq 0$ on $S \cap M^Q$;

(iv) the least integer $k$ such that there exist $k$ linear functionals $L_1, \ldots, L_k$ on $S$ with the property that $Q(x) \geq 0$ for all $x$ in $S$ satisfying the conditions $L_a(x) = 0$ ($a = 1, 2, \ldots, k$).

3. Results. The main purpose of this paper is to announce the results presented in this section. The details and more results are to appear elsewhere.

The definition of a singular conjugate point is found in Tomastik [7, p. 61] and Chellevold [1, p. 333]. It extends the definition of Morse and Leighton [5, p. 253], who treated the case $n = 1$. For $a \leq e \leq b$ let $A(e) = \{x \in A : x(t) = 0 \text{ for } e \leq t \leq b\}$, where $A$ is defined in §1 of this paper. Define the set $B$ in $A$ to be the union of the sets $A(e)$ for $a < e < b$. Observe that $B$ is actually a subspace of $A$.

**Theorem 3.1.** The following conditions are equivalent for some non-negative integer $k$:

(i) The signature of $J$ given by (1.3) on $B$ is $k$. 

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(ii) There is an \( \varepsilon_0 \) in \((a, b)\) such that \( \varepsilon_0 \leq \varepsilon < b \) implies that the signature of \( J \) given by (1.3) on \( A(\varepsilon) \) is \( k \).

(iii) The point \( a \) has exactly a finite number \( k \) of nonsingular conjugate points on \( a < t < b \).

(iv) The point \( b \) has exactly a finite number \( k \) of singular conjugate points on \( a < t < b \).

(v) \( b \) is not conjugate to \( b \).

Theorem 3.1 above contains Theorem 4.4, p. 337, of Chellevold [1].

Let \( U(t) \) be a conjugate system satisfying Euler’s equation

\[
\begin{align*}
\left[ R(t) \dot{U}(t) + Q^*(t)U(t) \right] &= \left[ Q(t) \dot{U}(t) + P(t)U(t) \right] \\
\text{and the conditions } U(a) &= 0, \quad \dot{U}(a) = I, \quad \det U(t) \neq 0 \text{ for } t \text{ near } b. \quad \text{Let us remark that there are } J’s \text{ which do not possess such conjugate systems.}
\end{align*}
\]

For \( y \in A \) and for \( t \) near \( b \) set

\[
S[y(t), a] = y^*(t)\left[ R(t) \dot{U}(t) + Q^*(t)U(t) \right] U^{-1}(t) \quad \text{for each } y \text{ in } D \text{ satisfying } \lim \inf_{t \to b} J(y) < \infty
\]

Let \( D \) be a subspace in \( A \) satisfying \( B \subseteq D \subseteq A \). The condition that \( \lim \inf_{t \to b} S[y(t), a] \geq 0 \) for each \( y \) in \( D \) satisfying \( \lim \inf_{t \to b} J(y) < \infty \) is called the singularity condition relative to \( D \) and belonging to \([a, b]\).

**THEOREM 3.2.** Assume that \( s(B) \) is finite. Let \( D \) be any subspace with \( B \subseteq D \subseteq A \). Let \( C \) be a subspace in \( B \) maximal relative to having \( J < 0 \). Let \( C^J = \{ x \in A : J(x, y) = 0 \text{ for all } y \in C \} \). The following conditions are equivalent:

(i) If \( x \) is in \( D \cap C^J \), then \( J(x) < \infty \) implies \( \lim \inf_{e \to b} S[x(e), a] \geq 0 \).

(ii) If \( x \) is in \( D \cap C^J \), then \( J(x) \geq 0 \).

(iii) The singularity condition relative to \( D \) holds; that is, if \( x \) is in \( D \), then \( J(x) < \infty \) implies \( \lim \inf_{e \to b} S[x(e), a] \geq 0 \).

**THEOREM 3.3.** Suppose that \( J(x, y) = \lim \inf_{e \to b} J_e(x, y) \) is bilinear on the subspace \( D \) where \( B \subseteq D \subseteq A \). Assume that \( s(B) \) is finite. Let \( C \) be a subspace in \( B \) maximal relative to having \( J < 0 \). Let \( C^J = \{ x \in A : J(x, y) = 0 \text{ for all } y \in C \} \). Then \( s(D) = s(B) \) if and only if \( x \) in \( C^J \cap D \) implies \( J(x) \geq 0 \).

**COROLLARY.** If \( J \) is bilinear on the subspace \( D \) with \( B \subseteq D \subseteq A \) and \( s(B) \) is finite, then \( s(D) = s(B) \) if and only if the singularity condition relative to \( D \) and belonging to \([a, b]\) holds.

The next theorem generalizes Theorems 2.3, 4.1, and 5.1 of Tomastik [7].

**THEOREM 3.4.** There is a subspace \( C \) of finite dimension \( k \) in \( B \) with \( C \) maximal relative to having \( J < 0 \) and \( J \geq 0 \) on \( C^J \cap D \) holds for a subspace
D with $B \subseteq D \subseteq A$ if and only if there are $k$ conjugate points to $b$ in $(a, b)$ and the singularity condition relative to $D$ and belonging to $[a, b]$ is satisfied.

**Corollary.** There is a subspace $C$ of finite dimension $k$ in $B$ with $C$ maximal relative to having $J < 0$ and $J \geq 0$ on $C^1$ holds if and only if there are $k$ conjugate points to $b$ in $(a, b)$ and the singularity condition relative to $A$ and belonging to $[a, b]$ is satisfied.

**Corollary.** For any subspace $D$ with $B \subseteq D \subseteq A$, $J \geq 0$ on $D$ holds if and only if there are no conjugate points to $b$ in $(a, b)$ and the singularity condition relative to $D$ and belonging to $[a, b]$ is satisfied.

**Bibliography**


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