Wiener’s classical tauberian theorem has been extended recently to some noncommutative, noncompact groups (see [1], [3], [8] and [10]). Our Theorems 1 and 2 are Wiener type theorems, and interest in them led to the study of contractible groups. It was rather surprising that all contractible Lie-groups are unipotent matrix groups (Theorem 3).

1. Contracting group extensions. A locally compact group $N$ is contractible provided it has enough contractions, i.e., for any compact set $K \subset N$ and any neighborhood $W$ of the identity in $N$, there is a homeomorphic automorphism $h \in \text{Aut } N$ such that $hK \subset W$. The ordered pairs $(K, W)$ form a directed set with respect to the relation $\leq$, defined by $(K, W) \leq (K', W')$ if and only if $K \subseteq K'$ and $W \supseteq W'$. For every $n = (K, W)$ choose a contraction $h_n$ with $h_nK \subset W$, then $\{h_n\}$ is a net and for any compact set $K \subset N$ we have $\lim_n h_nK = \{e\}$ ($e$ the neutral element of $N$).

A locally compact group $G$ is a contracting extension of its normal subgroup $N$ provided the set of restrictions to $N$ of inner automorphisms of $G$ contains enough contractions of $N$. Thus $N$ must be contractible to admit contracting extensions. For example, if $G' \subseteq \text{Aut } N$ is a locally compact group and contains enough contractions of $N$, then the semi-direct product $G = G' \ltimes N$ is a contracting extension of $N$.

If $G$ is an extension of $N$ and $G = G/N$ is the corresponding factor group we will usually denote their elements respectively by $x$, $\xi$, $\hat{x}$, their (left) Haar measures by $dx, d\xi, d\hat{x}$, and their moduli by $\Delta, \delta$ and $\Delta'$. We suppose that Weil’s formula $dx = d\xi d\hat{x}$ holds.

Let us suppose for a moment that $G$ is separable (i.e. has a countable basis of open sets). Then there exists a measurable cross-section $\sigma$ of $G$ with respect to $N$ (cf. [9]); i.e., there is a measurable function $\sigma: G' \to G$ with $\sigma(x) \in xN$ and $\sigma(e) = e$. Suppose further that there is a net $\{h_n\}$ of contractions of $N$ as above, such that $\lim_n h_n(x)$ exists for locally almost
all $x$ in $G$ (with respect to $dx$). Let $\sigma_n(x) = h_n(\sigma(x))$. $\sigma_n$ is then a measurable function and $\rho(x) = \lim_n \sigma_n(x)$ exists locally almost everywhere on $G$. Since $G$ is separable it is metrizable, and the net $\{h_n\}$ can be replaced by a sequence. By Egoroff's theorem (cf. [2]) we have the property:

(E) There is a measurable cross-section $\sigma$ of $G$ with respect to $N$ and a measurable function $\rho$ from $G$ into $G$; and for each compact set $K \subset G$ and every $\varepsilon > 0$, there is a compact set $K_1 \subset K$ such that $d\sigma(K \setminus K_1) < \varepsilon$ and the restrictions $\sigma_n|_{K_1}$ are continuous and converge uniformly to $\rho$ as functions on $K_1$.

From now on we will not use the separability of $G$ but we will suppose that the property (E) holds.

Let $L^1(G)$ be the set of all Haar-measurable and absolutely summable complex-valued functions on $G$. With the usual convolution and involution, $L^1(G)$ is an involutive Banach algebra, and $L^\infty(G)$ is its Banach space dual. $G$ acts weak-* continuously on $L^\infty(G)$ by the usual left and right translations. Subspaces which are closed under these actions are called bi-invariant.

An involutive Banach algebra $B$ is said to have the Wiener property if and only if:

(W) Every proper closed two-sided ideal $I \triangle B$ is contained in the kernel of an irreducible, continuous $*$-representation of $B$ on some Hilbert space.

$B$ is said to be tauberian if and only if it has the property:

(T) Every proper, closed two-sided ideal $I \triangle B$ is contained in a maximal modular two-sided ideal of $B$.

We will say that a group $G$ is tauberian (or has property (W)) if $L^1(G)$ is tauberian (or satisfies (W)).

**Theorem 1.** Let $G$ be a contracting extension of $N$ satisfying (E). If $G/N$ satisfies (W) or (T), then so does $G$.

The proof of this theorem is based on the following lemma and proposition. Since the canonical projection $p : G \to G$ is continuous and open and the function $\rho$ is measurable, the composite map $r = \rho \circ p : G \to G$ is measurable.

**Lemma 1.** Let $G$ be a contracting extension of $N$ satisfying (E), and let $M$ be a weak-* closed, bi-invariant subspace of $L^\infty(G)$. If $\phi \in M$ is left uniformly continuous on $G$ then $\phi \circ r \in M$.

**Proposition 1.** Let $G$ and $M$ be as in Lemma 1, and let $M_0$ be the subset of all $\phi \in M$ which are constant on the cosets with respect to $N$. Then $M_0$ is a nontrivial, bi-invariant subspace of $M$.

**Proposition 1' (Dual Version).** Let $G$ be as above. If $I$ is a proper,
closed two-sided ideal in \( L^1(G) \), and if \( J \) is the kernel of the morphism \( f \rightarrow f' \) of \( L^1(G) \) onto \( L^1(G) \) (where \( f'(x) = \int_N f(x \xi) d\xi \)), then the closure \( \text{cl}(I + J) \) is a proper, closed, two-sided ideal in \( L^1(G) \); equivalently the closure \( \text{cl}(I) \) of the image of \( I \) under the above morphism is a proper closed two-sided ideal in \( L^1(G) \).

2. Some extensions of contractible algebras. Let \( A \) be an involutive Banach algebra on which a locally compact group \( G \) acts strongly continuously by isometric, involutive, algebra automorphisms \( T_x \), \( x \in G \). The algebra \( A \) is \( T \)-contractible provided that there is a net \( \{x_n\} \) in \( G \) such that

(i) \( \lim_n(T_{x_n}a) \) exists in \( A \) for all \( a, b \in A \), and

(ii) for some \( u \in A \) the net \( \{T_{x_n}u\} \) is an approximating unit for \( A \).

For example, if \( N \) is a contractible group, \( A = \ell^1(N) \) and \( G = \text{Aut} \, N \) is a locally compact group, then \( A \) is \( T \)-contractible if we define \( T \) by

\[
T_x f(x) = \int_N f(xy) \, d\xi \cdot \xi, \quad x \in G, \; f \in A, \; \xi \in N.
\]

In fact \( T_{x_n}f \) converges to the scalar \( \lambda(f) = \int_N f(\xi) \, d\xi \). Since \( A \) contains approximating units, \( A \) can be isometrically imbedded in its adjoint algebra \( A^b \), which is itself an involutive Banach algebra with unit (cf. [7, §3]).

**Lemma 2.** Let \( A \) be a \( T \)-contractible algebra. The equation

\[
R_a b = \lim_n(T_{x_n}a)b \quad (a, b \in A)
\]

defines an involutive representation \( R \) of \( A \) into its adjoint algebra \( A^b \). The kernel \( j = \ker R \) of \( R \) is \( G \)-invariant, if the \( x_n \) belong to the center of \( G \).

Let \( L = L^1(G, A; T) \) be the generalized \( L^1 \)-algebra with trivial factor system (cf. [7, §1]). As a Banach space, \( L \) is isomorphic to the projective tensor product \( \ell^1(G) \otimes A \). The convolution of \( f, g \in L \) is defined by the Bochner integral

\[
f * g(x) = \int T_x f(xy) \cdot g(y^{-1}) \, dy,
\]

and the involution by \( f^*(x) = (T_{x^{-1}} f)(x) \). \( L \) can be viewed as an extension of the algebra \( A \) by the group \( G \) (cf. [4]).

Suppose \( j = \ker R \) is \( G \)-invariant. Let \( A^* = A/j \) be the involutive Banach algebra quotient of \( A \) by \( j \), and define \( T^* \) on \( A^* \) by \( T_x(a + j) = (T_x a) + j \). The canonical projection \( A \rightarrow A^* \) induces an isometric isomorphism \( L/J \cong L = L(G, A^*; T^*) \) which we denote (par abuse) by \( R^* \) (cf. [7, §5]). The kernel \( J \) of \( R^* \) can be identified with \( \ell^1(G) \otimes j \).

**Lemma 3.** Let \( A \) be a \( T \)-contractible algebra and assume that \( j = \ker R \) is \( G \)-invariant. Let \( J = \ker R^*_\star \) be as above.

(i) \( \lim_n(T_{x_n}f) * g = 0 \) for all \( f \in J \) and \( g \in L \), where \( (T_x f)(y) = T_x(f(y)) \).

(ii) Let \( p_i \) be an approximating unit of \( \ell^1(G) \); if \( R_u = \text{id}_A \) for some \( u \in A \) and \( p_i = T_{x_n}(p_i \otimes u) = p_i \otimes T_{x_n} u \), then \( \{p_i \} \) is an approximating unit of \( L \), where \( (i, n) \geq (i', n') \) iff \( i \geq i' \) and \( n \geq n' \).
PROPOSITION 2. Let $A$ be a $T$-contractible algebra and assume that $j = \ker R$ is $G$-invariant. If $I$ is a proper, closed, two-sided ideal in $L = L(G, A; T)$ then so is the closure of $I + J$.

By Proposition 2, $L$ will be wienerian $(W)$ or tauberian $(T)$ if $L$ has the respective property.

THEOREM 2. Let $A$ be a $T$-contractible algebra. Let $R$ be as in Lemma 1, but assume that each $R_a$ is a scalar multiple of the identity operator. Assume that $j = \ker R$ is $G$-invariant. If $G$ satisfies $(W)$ or $(T)$ then so does $L = L(G, A; T)$.

The method of proof in this paragraph is essentially the same as in [10], whereas the method in §1 is new, and different from the method in [3].

3. Contractible Lie groups and Lie algebras. A few facts about contractible groups in general are collected in

PROPOSITION 3. Let $G$ be a nontrivial contractible group.

(i) $G$ is neither compact nor discrete.

(ii) If $G$ is locally connected, then also globally.

(iii) If $G$ is locally simply connected, then also globally.

(iv) If $G$ has a nontrivial compact subgroup, then it has arbitrarily small ones.

(v) If $G$ has a compact open subset, then $G$ is totally disconnected.

Let $K$ be a nondiscrete, complete field of characteristic 0, and let $\lambda \to |\lambda|$ be a norm (= valuation) of $K$. Since $K$ is nondiscrete there are nonzero $\lambda_n \in K$ with $\lim_n |\lambda_n| = 0$. If $M \subset K$ is (norm-) bounded then the diameters of the sets $\lambda_n M$ converge to 0. Multiplication by a scalar $\lambda_n \neq 0$, defines an automorphism of $K$'s additive group. The additive group of $K$ is thus contractible if locally compact.

Let $\mathcal{G}$ be a finite-dimensional Lie algebra over $K$ with Lie product $(x, y) \to [x, y]$ and norm $x \to |x|$ for which $|[x, y]| \leq |x| \cdot |y|$. The norm $|h|$ of a Lie homomorphism $h$ of $\mathcal{G}$ is the norm of $h$ as a linear operator of the normed space $\mathcal{G}$; $|h| = \sup\{|hx|; |x| \leq 1\}$.

A contraction of the Lie algebra $\mathcal{G}$ is a Lie automorphism $h$ with $|h| < 1$. If $\mathcal{G}$ has one contraction $h$, then it has enough contractions and we call $\mathcal{G}$ contractible: the powers $h^n$ of $h$ map every bounded set eventually into any 0-neighborhood of $\mathcal{G}$, because their norms $|h^n|$ converge to 0.

PROPOSITION 4. Finite dimensional contractible Lie algebras over $K$ are nilpotent.

EXAMPLES. (1) All freely generated, nilpotent Lie algebras are contractible.
(2) All nilpotent Lie algebras of dimension \( \leq 6 \) are contractible, but some are not freely generated. (This last result is based on the classification of these Lie algebras in [11].)

A unipotent matrix over \( K \) is an (upper) triangular matrix of finite order with coefficients from \( K \) and 1's in the main diagonal. A unipotent group over \( K \) is (up to a global isomorphism) a group of unipotent matrices with matrix multiplication as its group operation, which is complete with respect to a norm topology on the respective matrix ring. The topology of a unipotent group does not depend on the choice of norm because \( K \) (etc.) is completely metrizable, and Baire's theorem applies.

**Proposition 5.** If \( \mathcal{G} \) is a finite dimensional nilpotent Lie algebra over \( K \) (not necessarily contractible) then \( \mathcal{G} \) can be imbedded into an associative matrix algebra \( A \) over \( K \), such that the power series \( \exp(x) = \sum_{n \geq 0} x^n/n! \), as evaluated in \( A \), reduces to a polynomial for all \( x \in \mathcal{G} \), and such that the global image \( \exp^\mathcal{G} \) of \( \mathcal{G} \) under \( \exp \) is a unipotent group.

The proof of this proposition depends on the theorems of Ado, Lie and Campbell-Hausdorff (cf. e.g. [5]).

**Theorem 3.** If \( G \) is a contractible Lie group of finite dimension over the field \( R \) of real numbers or the field \( \mathbb{Q}_p \) of \( p \)-adic numbers, then \( G \) is a unipotent group.

In the real case, the proof of Theorem 3 is achieved through Propositions 6 and 7 below, which in turn depend on classical theorems. In the \( p \)-adic case, however, we rely on results from [6], notably the "inversion of the Campbell-Hausdorff formula" [ibid., IV, 3.2.3].

**Proposition 6.** The Lie algebra \( \mathcal{G} \) of a contractible Lie group \( G \) over \( R \) is contractible and thus nilpotent.

**Proposition 7.** If \( G \) is a connected and simply connected nilpotent Lie group over \( R \) (not necessarily contractible), then \( G \) is a unipotent group.

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**References**


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