BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 6, November 1973

UNLINKING UP TO COBORDISM

BY M. A. GUTIÉRREZ

Communicated by Edgar Brown, April 10, 1973

0. **Introduction.** Let S_i be a copy of S^n ($n \geqq 3$) for $i = l, \ldots, m$. An m-link of dimension n is an embedding $L: S_1 + \cdots + S_m \to S^{n+2}$, where $+$ stands for disjoint union.

We say L is a boundary link if it extends to an embedding $V_1 + \cdots + V_m \to S^{n+2}$, where V_i is a $(n + 1)$-dimensional, compact, framed manifold with $\partial V_i = S_i$. The V_i are called *Seifert* manifolds for L. In particular, if the V_i are disks, we say that L is trivial.

A link L is split if we can find $(n + 1)$-spheres Σ_j ($j = 1, \ldots, m - 1$), smoothly embedded in S^{n+2} and disjoint from $\mathrm{Im}(L)$ as well as from each other, and such that each of the m connected components of $S^{n+2} - \bigcup \Sigma_j$ contains one of the knots $L(S_i)$.

In [2], the notion of cobordism is defined. The cobordism classes of m-links of dimension n form, under componentwise connected sum, an abelian group $C_n^{(m)}$. The group $C_n = C_n^{(1)}$ has been computed in [4] for n odd $\geqq 3$, and found to be trivial for n even $\geqq 2$ in [2]. The purpose of this note is to announce the following result:

Every m-link of dimension n $\geqq$ 3 is cobordant to a split link; in particular:
If n is odd $\geqq 3$, $C_n^{(m)} = C_n \oplus \cdots \oplus C_n$ (m times).
If n is even $\geqq 4$, $C_n^{(m)} = 0$.

1. **The fundamental group.** The normal bundle of $\mathrm{Im}(L) \subset S^{n+2}$ is trivial; let

$$X = \overline{S^{n+2} - (T_1 + \cdots + T_m)}$$

where T_i is a tubular neighborhood of $L(S_i)$ diffeomorphic to $S_i \times D^2$. The compact manifold X is called the *complement* of L and $\pi = \pi_1(X)$ its *group*. Observe, $\partial X = (S_1 \times S^1) + \cdots + (S_m \times S^1)$.

The inclusion $\partial X \subset X$ induces a homomorphism of fundamental groups $h: F_m \to \pi$, where F_m is the free group in m generators a_i. The elements $h(a_i)$ are called *meridians* of L.

LEMMA 1. *The homomorphism h induces a monomorphism*

$$h_*: F_m \to \pi/\pi_\omega$$

where π_ω *is the* ω*th term of the lower central series of* π (*cf.* [6, *p.* 157]).

AMS (MOS) subject classifications (1970). Primary 57C45.

Key words and phrases. Boundary link, split link, lower central series.

LEMMA 2. *The link L is boundary if, and only if, h_* is an isomorphism.*

For proofs of these statements see [**1**].

2. **Cobordism.** In [**1**] we prove:

LEMMA 3. *Every boundary link of dimension $\geqq 3$ is cobordant to a split link.*

Thus, our announced result will be established with the following:

LEMMA 4. *Every m-link of dimension $\geqq 3$ is cobordant to a boundary link.*

SKETCH OF PROOF FOR $m = 2$. Let $L: S_1 + S_2 \to S^{n+2}$ be an arbitrary link with group π. By [**2**, Theorem I.1], we can construct a null-cobordant link L_0 with group π; let $h: S_1 \times I + S_2 \times I \to S^{n+2} \times I$ be a cobordism between L_0 and the trivial link. By a Thom construction, it is possible to extend h to an embedding h' of a manifold W^{n+2} transverse at $\partial(S^{n+2} \times I)$ and with boundary $V \cup (S_1 \times I + S_2 \times I) \cup V'$, where,

(i) $h'(W) \cap S^{n+2} \times \{0\} = V$ and $\partial V = \mathrm{Im}(L_0)$,

(ii) $h'(W) \cap S^{n+2} \times \{1\} = V'$ and $\partial V' = S^n \times I$.

Naturally, W need not be 1-connected but if we assume $S^{n+2} \times I \subset \partial(D^{n+3} \times I)$, we can, in the fashion of [**2**, Chapter III], add embedded 2-handles to $h'(W)$ to make it simply connected. Let U^{n+3} be the trace of the surgeries. As in [**4**], $\partial U = W \cup (\partial W \times I) \cup W'$, where W' is 1-connected and with connected boundary $\partial W \times \{1\}$. Since W' is $(n+1)$-collapsible, we can engulf it by a $(n+4)$-cell in $D^{n+3} \times I$. We obtain a cobordism $\bar{h}$ between L_0 and the trivial link that extends to an embedding $\bar{h}'$ of a simply connected manifold W', transversal at $\partial(S^{n+2} \times I)$ and with $\partial W' \cong \partial W$.

If $Z = S^{n+2} \times I - \mathrm{Im}(\bar{h}')$, by the Van Kampen theorem,

$$\pi_1(S^{n+2} \times I - \mathrm{Im}(\bar{h})) = \pi_1(Z) * \mathbf{Z}a_1,$$

where a_1 is a meridian in L_0.

The map $\bar{h}$ can be slightly altered so that the projection $s: S^{n+2} \times I \to I$ is a *nice* Morse function when restricted to $\mathrm{Im}(\bar{h})$. By these means we find an embedded handle decomposition [**5**, §2] of the cylinders $S_i \times I$ in $S^{n+2} \times I$. We can rearrange the embedded handles by index [loc. cit., Lemma 3]. It turns out that if there are r 0-handles (i.e. r local minima for $s \mid \mathrm{Im}(\bar{h})$), there exist r 1-handles (i.e. saddle points) with critical values $t_1 < \cdots < t_r$, cancelling the 0-handles. If $\varepsilon > 0$ is sufficiently small, the space $s^{-1}(t_r + \varepsilon) = X^*$ is the complement of a link L^*- cobordant, of course, to L_0- and with fundamental group π^* so that π^*/π^*_ω is isomorphic to a subgroup of $\pi_1(Z) * \mathbf{Z}a_1$ of the form $G * \mathbf{Z}a_1$. That L^* is a boundary link follows easily from Lemma 2.

Notice that, in the process, we have proved the following algebraic lemma:

Let π be a link group; for some integer r, if L is the free group in the letters $b_1, \ldots, b_r$, there exist relations

(R) $a_{i_j} w_j b_j w_j^{-1} = 1$, *with* $w_j \in \pi * L$ *and* a_{i_j} *a meridian in* π,

such that the group $\pi * L$ with relations (R) maps onto the free group generated by the meridians a_i.

If t_0 is a critical value of $s \mid \mathrm{Im}(\bar{h})$ of index 0 and $\varepsilon > 0$ is small, $s^{-1}(t_0 + \varepsilon)$ is obtained from $s^{-1}(t_0 - \varepsilon)$ by attaching a 1-handle. Similarly, if t_0 is a critical value of index 1, $s^{-1}(t_0 + \varepsilon)$ is obtained by attaching a 2-handle (along the word $a_{i_j} w_j b_j w_j^{-1}$, where a_{ij} is a meridian and b_j is the core of one of the attached 1-handles) to $s^{-1}(t_0 - \varepsilon)$.

As a corollary, a null cobordism for L_0 is attained by attaching r 1-handles and r 2-handles to the complement of L_0. Similarly, if X is the complement of our arbitrary link L, perform the same surgeries on $X \times I$ to obtain a link L' cobordant to L and with a fundamental group that maps epimorphically onto the free group generated by the meridians a_j. This completes the proof of Lemma 4.

It is quite disappointing that a purely algebraic proof of the lemma above could not be found. Conceivably, this proof should be based on the results of Baumslag and Lazard.

One of the main corollaries of our results is the description and characterization of the Alexander polynomials of a link; let L be a link and X its complement. Since $\pi_1(X)$ abelianizes to $\boldsymbol{Z}^m$, the universal abelian cover $\tilde{X}$ of X has a natural action of $\boldsymbol{Z}^m$ on it and its qth dimensional homology $H_q(\tilde{X}; Q)$ is then a Γ_m-module, where Γ_m is the rational group ring of $\boldsymbol{Z}^m$, the Laurent polynomials in m variables with rational coefficients. As in [**6**, p. 117] we can define the elementary invariants of $H_q(\tilde{X}; Q)$. These are the Alexander polynomials as defined in [**3**]. If L is a boundary link, the techniques of [**3**] allow us to describe the polynomials; however, since the universal abelian cover of the complement of any link is obtained by doing 1- and 2-surgeries to the cover of the complement of a boundary link, the description found in the particular case holds in general. (Special arguments are needed for $q = 1, 2$.)

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DEPARTMENT OF MATHEMATICS, QUEENS COLLEGE, CITY UNIVERSITY OF NEW YORK, FLUSHING, NEW YORK 11367