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THE SINGULARITIES OF THE $\mathcal{S}$-MATRIX AND GREEN'S FUNCTION ASSOCIATED WITH PERTURBATIONS OF $-\Delta$ ACTING IN A CYLINDER

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It is the purpose of this note to study the singularities of the $\mathcal{S}$-matrix and Green's function associated with the operators considered in [1]–[3]. As will be seen, there are a countable number of branch points, as well as a countable number of different $\mathcal{S}$-matrices associated with these operators. In this respect, these results differ considerably from those drawn from quantum mechanical scattering[2] and the exterior problem (see e.g. [**4**] and [**5**]).

1. **Preliminaries.** Let S denote the semi-infinite cylinder in R^N, N-dimensional Euclidean space ($N \geqq 2$), with arbitrary bounded, smooth $N-1$ dimensional cross-section l. Thus S consists of the points $x = ((x_1, \ldots, x_{N-1}), x_N) = (\tilde{x}, x_N)$, where $\tilde{x} \in l$ and $x_N \geqq 0$.[3] Let Ω denote the domain with smooth boundary $\dot{\Omega}$, obtained from S by perturbing a finite part of $\dot{S}$. Thus $\Omega = S$ for $x_N \geqq \mathring{x}_N$ for some fixed $\mathring{x}_N > 0$.

We now define the operators $A_0(A)$ by $-\Delta$ acting in $L_2(S)$ ($L_2(\Omega)$) and associated with zero Dirichlet boundary conditions on $\dot{S}(\dot{\Omega})$. Let A_l denote the corresponding operator defined in $L_2(l)$ and let $\{v_n\}$ and $\eta_n(\tilde{x})$ denote a complete set of eigenvalues (in increasing order) and corresponding orthonormal eigenfunctions for A_l. Let A^c denote that part of A orthogonal to all of its eigenvalues, Λ denote the set of eigenvalues of A and $\Lambda' = \Lambda \cup \{v_n\}$.

It was shown in [**1**] that a complete set of generalized eigenfunctions for A_0 and A^c are given by

$$w_n^0(x; \lambda) = (2/\pi)^{1/2} \sin(\lambda - v_n)^{1/2} x_N \eta_n(\tilde{x}), \qquad \lambda \notin \{v_n\},$$

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[1] Work performed under the joint auspices of the Atomic Energy Commission and the Mathematics Research Center at the University of Wisconsin.

[2] Our results are related to wave progagation in a waveguide.

[3] We might just as easily consider the infinite cylinder, $S' = (x = (\tilde{x}, x_N) \mid \tilde{x} \in l, -\infty < x_N < \infty)$.

and

$$w_n^-(x;\lambda) = w_n^0(x;\lambda) + v_n^-(x;\lambda), \qquad \lambda \notin \Lambda',$$

where

$$(1.1)\qquad \begin{aligned} v_n^-(x;\lambda) = &\sum_{n'=1}^{m} c_{n'}^n(\lambda)\exp\{-i(\lambda - \nu_{n'})^{1/2}\, x_N\}\eta_{n'}(\tilde{x}) \\ &+ \sum_{n'=m+1}^{\infty} c_{n'}^{n-}(\lambda)\exp\{-(\nu_{n'} - \lambda)^{1/2}x_N\}\eta_{n'}(\tilde{x}) \end{aligned}$$

for $x_N \geqq \mathring{x}_N$ and $\lambda \in (\nu_m, \nu_{m+1})$, $(\ )^{1/2}$ denoting the positive square root. Another complete set of generalized eigenfunctions, $w_n^+(x;\lambda) = w_n^0(x;\lambda) + v_n^+(x;\lambda)$, for A^c are defined analogously with $\exp\{-i(\ )^{1/2}\}$ replaced by $\exp\{i(\ )^{1/2}\}$ and $c_{n'}^{n-}(\lambda)$ by $c_{n'}^{n+}(\lambda)$.

It was proven in [3] that the $\mathcal{S}$-matrix, $\mathcal{S}_m(\lambda)$, associated with A_0 and A at the point $\lambda \in (\nu_m, \nu_{m+1})$, $\lambda \notin \Lambda$, is given by the matrix

$$(1.2)\qquad \mathcal{S}_m(\lambda) = I_m + T_m(\lambda),$$

where I_m is the identity matrix of rank m and

$$(1.3)\qquad \begin{aligned} &T_m(\lambda) = (t_{n,n'}(\lambda)) \quad \text{with} \\ &t_{n,n'}(\lambda) = -(2\pi)^{1/2} i c_{n'}^{n-}(\lambda), \quad n, n' = 1, \ldots, m. \end{aligned}$$

Note that the rank m of $\mathcal{S}_m(\lambda)$ varies with λ.

It follows from the arguments of [6] that Green's function, $G_0^-(x, y; \lambda)$, for the operator $A_0 - \lambda$ is given by

$$(1.4)\qquad \begin{aligned} G_0^-(x, y; \lambda) &= \sum_{n\neq 1}^{\infty} \frac{\sin(\lambda - \nu_n)^{1/2}x_N \exp\{-i(\lambda - \nu_n)^{1/2}y_N\}}{(\lambda - \nu_n)^{1/2}}\, \eta_n(\tilde{x})\eta_n(\tilde{y}), \\ &\qquad\qquad \text{for } x_N < y_N \\ &= \sum_{n=1}^{\infty} \frac{\exp\{-i(\lambda - \nu_n)^{1/2}x_N\}}{(\lambda - \nu_n)^{1/2}} \sin(\lambda - \nu_n)^{1/2} y_N \eta_n(\tilde{x})\eta_n(\tilde{y}), \\ &\qquad\qquad \text{for } x_N > y_N, \end{aligned}$$

where $x, y \in S$ and $\operatorname{Im}(\lambda - \nu_n)^{1/2} < 0$, $n = 1, 2, \ldots$. In view of (1.1) and (1.4), it is natural to define the infinitely sheeted Riemann surface R_∞, obtained by making each point ν_n a branch point of order one. By $\Gamma_{n_1,\ldots,n_k}$ ($\mathrm{cl}(\Gamma_{n_1,\ldots,n_k})$), we shall mean that sheet of R_∞ consisting of those points λ for which $0 < \arg(\lambda - \nu_n) < 2\pi$ ($0 \leqq \arg(\lambda - \nu_n) < 2\pi$) for $n = n_1, \ldots, n_k$ and $-2\pi < \arg(\lambda - \nu_n) < 0$ ($-2\pi \leqq \arg(\lambda - \nu_n) < 0$) for all remaining n. The "physical sheet", Γ_0 ($\mathrm{cl}(\Gamma_0)$), shall consist of those λ satisfying $-2\pi < \arg(\lambda - \nu_n) < 0$ ($-2\pi \leqq \arg(\lambda - \nu_n) < 0$), $n = 1, 2, \ldots$.

It can be easily seen that $G_0^-(x, y; \lambda)$, defined initially on Γ_0 has an analytic continuation onto all of R_∞ in the following sense. Consider

Γ_m, $m \geqq 1$, suppose $\mathscr{G} = [a, b] \subset (v_J, v_{J+1})$ for an arbitrary $J \geqq m$ and set $\kappa = (\lambda - v_m)^{1/2}$ for each $\lambda \in \mathscr{G}$. Then the function $\tilde{G}_0^-(x, y; \kappa) \equiv_{\mathrm{df}} G_0^-(x, y; \lambda)$ is an analytic function of κ for each $\kappa \in \operatorname{Im} \kappa < 0 \cup [(a - v_m)^{1/2}, (b - v_m)^{1/2}] \cup \operatorname{Im} \kappa > 0$ such that $\operatorname{Im}(\kappa^2 + v_m - v_n)^{1/2} \leqq 0$ for $n \neq m$. Hence $G_0^-(x, y; \lambda)$ has an analytic continuation from Γ_0 onto Γ_m across $\mathscr{G}$. We define an analytic (meromorphic) continuation of a complex or operator-valued function $F(\lambda)$ from an arbitrary sheet of R_∞ onto any other sheet in an analogous fashion. If such an analytic (meromorphic) continuation exists for each sheet of R_∞, we say that $F(\lambda)$ is analytic (meromorphic) on R_∞. We shall say that $\lambda_0 \in R_\infty$ is a pole of $F(\lambda)$ if $\kappa_0 = (\lambda_0 - v_m)^{1/2}$ is a pole of $\tilde{F}(\kappa) = F(\kappa^2 + v_m)$ corresponding to any of the countably many possible continuations described above.

2. **Meromorphic continuations.** Let $G^-(x, y; \lambda)$ denote Green's function for the operator $A - \lambda$, where $x, y \in \Omega$ and $\lambda \in \Gamma_0$.

THEOREM 1. (a) *$G^-(x, y; \lambda)$ has a meromorphic continuation from Γ_0 onto all of R_∞.* (b) *Suppose $\mathscr{G} \subset (v_m, v_{m+1}) - \Lambda$. Then $\mathscr{S}_m(\lambda)$ and each $w_n^-(x; \lambda)$, $n = 1, \ldots, m$, has a meromorphic continuation from $\mathscr{G}$ onto each sheet, $\Gamma_{n_1,\ldots,n_k}$, of R_∞ across (v_m, v_{m+1}), provided $0 \leqq n_1, \ldots, n_k \leqq m$.*

We shall outline the proof of Theorem 1 as follows. Set $\gamma = \dot{\Omega} - \dot{\Omega} \cap \dot{S}$, $\bar{\gamma}$ = closure of γ and $B = C(\bar{\gamma})$. Thus $\eta(x) \in B$ if $\eta(x)$ is a continuous function defined on $\bar{\gamma}$. We set $\|\eta\|_B = \max_{x \in \bar{\gamma}} |\eta(x)|$. Note that $\bar{\gamma}$ is compact by our definition of Ω. We define the integral operator T_λ by

$$T_\lambda \eta(x) = 2 \int_{\bar{\gamma}} \eta(y) \frac{\partial}{\partial v_y} G_0^-(x, y; \lambda)\, dS_y{}^4$$

for each $\eta(x) \in B$, $x \in \bar{\gamma}$ and $\lambda \in R_\infty$.

LEMMA 1. *T_λ is a compact, analytic B-valued function of λ and $\mathscr{T}_\lambda \equiv_{\mathrm{df}} (T_\lambda - I)^{-1}$ is a mermorphic B-valued function of λ on R_∞.*

Lemma 1 is the key result needed in the proof of Theorem 1 and follows employing the methods of potential theory as well as a result of Steinberg, [7, Theorem 1]. We denote the poles of $\mathscr{T}_\lambda$ on R_∞ by $\mathscr{D}$. Set

$$\hat{T}_\lambda \eta(x) = 2 \int_{\bar{\gamma}} \eta(y) \frac{\partial G_0^-(x, y; \lambda)}{\partial vy}\, dSy$$

for each $\eta(x) \in B$, $\lambda \in R_\infty$ and $x \in \Omega$.

[4] $G_0^-(x, y; \lambda)$ is defined in $\Omega - S$ by (1.4) with each $\eta_n(\tilde{x})$ continued across $\dot{l}$ as an odd function.

LEMMA 2. *For each* $x, y \in \Omega$ $(x \neq y)$ *and* $\lambda \in R_\infty - \mathscr{D}$, *we have*

$$G^-(x, y; \lambda) = G_0^-(x, y; \lambda) + \hat{T}_\lambda(\eta_\lambda(\cdot, y))(x),$$

where $\eta_\lambda(\chi', y) = -\mathscr{T}_\lambda(G_0^-(\cdot, y; \lambda))(x')$, $x' \in \gamma$.

Lemma 2 follows from the properties of $G_0^-(x, y; \lambda)$ as well as results from potential theory. An analogue of Lemma 2 follows for each $w_n^-(x; \lambda)$ in the same way. This combined with Lemma 1 and (1.1)–(1.3) implies Theorem 1. The poles of each of the functions $w_n^-(x; \lambda)$, $\mathscr{S}_m(\lambda)$ and $G^-(x, y; \lambda)$ belong to $\mathscr{D}$. We remark that we can also derive the meromorphic continuation of $\mathscr{S}_m(\lambda)$ in an easier way without employing the operator T_λ. The detailed proofs of all of the results of this note will appear elsewhere.

3. **Resonant states.** We now characterize the poles of $\mathscr{S}_m(\lambda)$ in terms of "resonant states".

DEFINITION. Suppose that m is a fixed positive integer, $\lambda_0 \in \Gamma_{n_1,\ldots,n_k}$, $1 \leqq n_1, \ldots, n_k \leqq m$, and there exists a nontrivial solution, $w(x; \bar{\lambda}_0)$, of

$$(3.1) \qquad (\Delta + \bar{\lambda}_0)w(x; \bar{\lambda}_0) = 0 \quad \text{in } \Omega, \qquad w(x; \bar{\lambda}_0) = 0 \quad \text{on } \dot{\Omega},$$

where $\bar{\lambda}_0$ denotes that value of $\bar{\lambda}_0$ in $\Gamma_{n_1,\ldots,n_{k-1}}$. Suppose also that there exist constants c_n, $n = 1, 2, \ldots,$ with some $c_j \neq 0$, $1 \leqq j \leqq m$, $j \neq n_1, \ldots, n_{k-1}$ such that

$$(3.2) \qquad \begin{aligned} w(x; \bar{\lambda}_0) = {} & \sum_{n=1}^{m} c_n \exp\{i(\bar{\lambda}_0 - \nu_n)^{1/2} x_N\} \eta_n(\tilde{x}) \\ & + \sum_{n=m+1}^{\infty} c_n \exp\{-i(\bar{\lambda}_0 - \nu_n)^{1/2} x_N\} \eta_n(\tilde{x}) \end{aligned}$$

for $x_N \geqq \mathring{x}_N$. Then we shall say that λ_0 is a $\Gamma_{n_1,\ldots,n_k}$ resonant state for $A_{\mathscr{G}}$ ($\mathscr{G}$ as in Theorem 1).

Note that $w(x; \bar{\lambda}_0)$ is exponentially blowing up for x_N large.

THEOREM 2. *Let* $\Gamma_{n_1,\ldots,n_k}$ *denote an arbitrary sheet of* R_∞ *such that* $1 \leqq n_1, \ldots, n_k \leqq m$. *Suppose that* $\mathscr{G}$ *is the interval of Theorem* 1(b), $\lambda_0 \in \Gamma_{n_1,\ldots,n_k}$ *and* $\bar{\lambda}_0$ *is chosen so that* $\bar{\lambda}_0 \in \Gamma_{n_1,\ldots,n_{k-1}}$. *Then* λ_0 *is a pole of* $\mathscr{S}_m(\lambda)$ *if and only if* λ_0 *is a* $\Gamma_{n_1,\ldots,n_k}$ *resonant state for* $A_{\mathscr{G}}$.

Theorem 2 is proved by obtaining explicit formulas relating the resonant states and the $\mathscr{S}_m(\lambda)$, employing the techniques of [**8**, §3]. In a future publication, we shall give concrete examples of resonant states. These will be obtained from the theory of waveguides.

4. **Perturbations due to a potential.** Now suppose that A is the operator given by $-\Delta + q(x)\cdot$ associated with the zero Dirichlet boundary

condition in S. It will be proved elsewhere, again employing the integral equation method, that analogues of Theorems 1 and 2 hold for A_0 and A, provided the real-valued potential $q(x)$ satisfies the condition: (C) $q(x) \in L_2 \operatorname{loc}(S)$ and $|q(x)| \leqq Ke^{-\alpha|x_N|}$ for $x_N \geqq \mathring{x}_N$ and positive constants K and α.

In this case the meromorphic continuations onto the sheet $\Gamma_{n_1,\ldots,n_k}$ are only valid in the intersection, $\bigcap_{j=1}^k M_{n_j}$, of the strips $M_{n_j} \equiv_{\mathrm{df}} \{\lambda | \, |\operatorname{Im}(\lambda - \nu_{n_j})^{1/2}| < \alpha/2\}$. Furthermore, equation (3.2) is replaced by an asymptotic relation of the same form and similarly for the radiation condition (1.1). In the special case in which $q(x) = q(x_N)$, the study of resonant states may be readily replaced by the corresponding problem for the one-dimensional Schroedinger operator on the interval $[0, \infty)$.

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