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AUTOMORPHISMS OF THE LATTICE OF RECURSIVELY ENUMERABLE SETS

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Let $\mathscr{E}$ denote the lattice of recursively enumerable (r.e.) sets under inclusion, and let $\mathscr{E}^*$ denote the quotient lattice of $\mathscr{E}$ modulo the ideal $\mathscr{F}$ of finite sets. For $A \in \mathscr{E}$ let A^* denote the equivalence class in $\mathscr{E}^*$ which contains A. An r.e. set A is *maximal* if A^* is a coatom (maximal element) of $\mathscr{E}^*$. Let Aut $\mathscr{E}$ (Aut $\mathscr{E}^*$) denote the group of automorphisms of $\mathscr{E}$ ($\mathscr{E}^*$). We prove that, for any two maximal sets A and B, there exists $\Phi \in \operatorname{Aut} \mathscr{E}$ such that $\Phi(A)=B$. It follows that for each $k \geqq 1$ the group Aut $\mathscr{E}^*$ is k-ply transitive on its coatoms. This demonstrates much more uniformity of structure of $\mathscr{E}$ than was supposed, and answers a question of Martin and Lachlan [**1**, p. 36]. We also use automorphisms to relate the *structure* of an r.e. set to its *degree*, particularly for degrees $\boldsymbol{d}$ which are *high* ($\boldsymbol{d}'=\boldsymbol{0}''$) or *low* ($\boldsymbol{d}'=\boldsymbol{0}'$), and as corollaries we answer questions and extend results of Lachlan, Martin, Sacks, Yates, and others. The proofs involve infinite-injury priority arguments like those of Sacks [**11**], [**12**], and [**13**], but here an altogether different method is needed to resolve conflicts between opposing requirements. The numbering of results in §1 and §2 corresponds to that of [**15**] where full proofs will appear. The results in §3 will appear in [**16**] and [**17**].

1. **Background information.** For $A, B \in \mathscr{E}$, let $A \equiv_{\mathscr{E}} B$ ($A^* \equiv_{\mathscr{E}^*} B^*$) denote that there exists $\Phi \in \operatorname{Aut} \mathscr{E}$ (Aut $\mathscr{E}^*$) such that $\Phi(A)=B$ ($\Phi(A^*)=B^*$). A permutation p of N *induces* an automorphism Φ of $\mathscr{E}$ ($\mathscr{E}^*$) if

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for all $W \in \mathscr{E}$, $\Phi(W)=p(W)$ $(\Phi(W^*)=(p(W))^*)$. Note that every recursive permutation induces some $\Phi \in \operatorname{Aut} \mathscr{E}$, and every $\Phi \in \operatorname{Aut} \mathscr{E}$ induces some $\Psi \in \operatorname{Aut} \mathscr{E}$. The following results answer questions of Rogers [**10**, pp. 228–229].

THEOREM 1.1 (LACHLAN). *There are $2^{\aleph_0}$ automorphisms of $\mathscr{E}^*$.*

COROLLARY 1.2. *Not every $\Phi \in \operatorname{Aut}(\mathscr{E}^*)$ is induced by a recursive permutation of N.*

THEOREM 1.3. *Every $\Phi \in \operatorname{Aut}(\mathscr{E}^*)$ is induced by some permutation of N*

COROLLARY 1.4. *Every $\Phi \in \operatorname{Aut}(\mathscr{E}^*)$ is induced by some $\Psi \in \operatorname{Aut}(\mathscr{E})$.*

COROLLARY 1.5. *If $A, B \in \mathscr{E}$ are infinite and coinfinite then $A \equiv_{\mathscr{E}} B$ if and only if $A^* \equiv_{\mathscr{E}^*} B^*$.*

Thus, a property of r.e. sets which are well defined on $\mathscr{E}^*$ is invariant under $\operatorname{Aut} \mathscr{E}^*$ just if it is invariant under $\operatorname{Aut} \mathscr{E}$ in which case it is called *invariant*. By these corollaries from now on we can use $\mathscr{E}$ and $\mathscr{E}^*$ "interchangeably" selecting whichever is more convenient. In Lachlan's method (Theorem 1.1) $\Phi \in \operatorname{Aut} \mathscr{E}^*$ is induced by a permutation p which is obtained by "piecing together" recursive permutations in a nonrecursive fashion, but no new elements are produced in the *$\mathscr{E}$-orbit* of $A \in \mathscr{E}$, namely the equivalence class $\{B: A \equiv_{\mathscr{E}} B\}$. To do this requires a more effective construction such as that of Martin, which can be used to prove the noninvariance of hypersimplicity (Theorem 1.9), creativeness (Theorem 1.10), and Turing degree. Martin's idea is to induce $\Phi \in \operatorname{Aut} \mathscr{E}$ by a permutation which although not recursive is the *limit* of recursive permutations. To insure that p induces an automorphism, one attempts to interchange elements only if they have the same "e-state".

2. **Automorphisms and maximal sets.** Let $\{W_n\}_{n \in N}$ be an acceptable numbering of the r.e. sets [**10**, p. 41]. We call $\Phi \in \operatorname{Aut} \mathscr{E}^*$ *effective* if there is some recursive permutation h of N such that $\Phi(W_n)=W_{h(n)}$ for all n. All $\Phi \in \operatorname{Aut} \mathscr{E}^*$ produced by Martin's "finite-injury" method (Theorem 1.9) are effective, roughly because one has such strong control over the inducing permutation p and the two sets being constructed. Effective automorphisms, however, will not suffice for the maximal set case, as we discovered after Lachlan suggested that there might be no *uniform* way to prove $A \equiv_{\mathscr{E}} B$ given maximal sets A and B.

THEOREM 2.1. *There exist maximal sets A and B such that $\Phi(A^*) \neq B^*$ for every effective $\Phi \in \operatorname{Aut} \mathscr{E}^*$.*

On the other hand our first principal result is that noneffective automorphisms will suffice.

THEOREM 2.3. *If A and B are maximal sets then $A \equiv_{\mathscr{E}} B$.*

We call $A \in \mathscr{E}$ *quasimaximal of rank n* if A is the intersection of n maximal sets M_i, $1 \leqq i \leqq n$, whose complements are pairwise disjoint. By applying Theorem 2.3 to each of n disjoint recursive sets which separate the $(M_i)^{-}$, $1 \leqq i \leqq n$, we see that $A \equiv_{\mathscr{E}} B$ if A and B are quasimaximal of the same rank (Corollary 2.6). Hence, for all $k \geqq 1$, the group Aut $\mathscr{E}^*$ is k-ply transitive on the coatoms of $\mathscr{E}^*$ (Corollary 2.7).

A class $\mathscr{C} \subseteq \mathscr{E}$ is a *skeleton* if $\mathscr{C}^* = \mathscr{E}^*$ where $\mathscr{C}^* = \{W^* : W \in \mathscr{C}\}$. To prove Theorem 2.3 we first choose appropriate skeletons $\{U_n\}_{n \in N}$ and $\{V_n\}_{n \in N}$ depending upon A and B, respectively. To specify the automorphism we give a permutation p of N and recursive functions f and g such that for all n, $W^*_{f(n)} = (p(U_n))^*$ and $W_{g(n)} = (p^{-1}(V_n))^*$. The heart of the method is a difficult result called the Extension Theorem (Theorem 2.2) which gives certain sufficient conditions under which a partial mapping on $\mathscr{E}$ can be extended to an automorphism. The proof is presented in the informal style of Lerman's "pinball machines" [5] to give the reader a clear picture of the dynamics of the construction.

3. **The structure of an r.e. set and its degree.** Recursion theorists have been interested in the relationship between the structure of an r.e. set A and its degree, denoted by deg A, ever since Post [8] asked for a simple property on the complement of $A \in \mathscr{E}$ which guarantees incompleteness, i.e., $\mathbf{0} < \deg A < \mathbf{0}'$. The existence of such a property remains an open question [11, $Q(3)$, p. 172], although we give a partial answer by showing that no such property can be lattice-invariant, as are the properties of simplicity, hyper-hypersimplicity, and maximality.

THEOREM 3.1. *For any nonrecursive $A \in \mathscr{E}$ there exists $B \in \mathscr{E}$ of degree $\mathbf{0}'$ such that $A \equiv_{\mathscr{E}} B$.*

A corollary is Yates result [18] that there is a complete maximal set.

Let $\boldsymbol{R}$ denote the class of r.e. degrees. For each $n \geqq 0$ define the subclasses of $\boldsymbol{R}$,

$$\boldsymbol{H}_n = \{\boldsymbol{d} : \boldsymbol{d} \text{ r.e. and } \boldsymbol{d}^{(n)} = \mathbf{0}^{(n+1)}\}, \quad \text{and}$$

$$\boldsymbol{L}_n = \{\boldsymbol{d} : \boldsymbol{d} \text{ r.e. and } \boldsymbol{d}^{(n)} = \mathbf{0}^{(n)}\},$$

where $\boldsymbol{d}^{(0)} = \boldsymbol{d}$. It is well known [10, pp. 290–294] that for each n, $\boldsymbol{H}_n \subsetneqq \boldsymbol{H}_{n+1}$ and $\boldsymbol{L}_n \subsetneqq \boldsymbol{L}_{n+1}$ and that there exists an r.e. degree $\boldsymbol{d}$ such that for all n, $\boldsymbol{d} \notin \boldsymbol{H}_n \cup \boldsymbol{L}_n$. The degrees in $\boldsymbol{H}_1$ ($\boldsymbol{L}_1$) are called *high* (*low*). (This terminology is often used with the condition "$\boldsymbol{d}$ r.e." above replaced by the weaker condition "$\boldsymbol{d} \leqq \mathbf{0}'$".)

A class $\boldsymbol{C} \subseteq \boldsymbol{R}$ is called *$\mathscr{E}$-definable* if $\boldsymbol{C} = \{\deg W : W \in \mathscr{C}\}$ for some lattice-invariant class $\mathscr{C} \subseteq \mathscr{E}$. For example, let $\boldsymbol{M}$ denote the class of degrees of maximal sets, and $\boldsymbol{A}$ the class of degrees of atomless sets, that is coinfinite r.e. sets which have no maximal supersets. The beautiful theorems of Martin [7] assert that $\boldsymbol{M} = \boldsymbol{H}_1$ and $\boldsymbol{A} \supseteq \boldsymbol{H}_1$. The difficult parts of Martin's results, namely that $\boldsymbol{M} \supseteq \boldsymbol{H}_1$ and $\boldsymbol{A} \supseteq \boldsymbol{H}_1$, can be obtained from the following result [**16**] suggested by Carl Jockusch. For $A \in \mathscr{E}$ define $\mathscr{L}(A) = \{W : W \in \mathscr{E} \,\&\, W \supseteq A\}$. Note that $\mathscr{L}(A)$ is a lattice under inclusion.

THEOREM 3.2. *For any nonrecursive $A \in \mathscr{E}$ and any $\boldsymbol{d} \in \boldsymbol{H}_1$ there exists an r.e. $B \in \boldsymbol{d}$ such that $\mathscr{L}(A) \simeq \mathscr{L}(B)$.*

Thus, no isomorphism-invariant property on $\mathscr{L}(A)$ can guarantee either completeness or incompleteness. Another corollary is Lachlan's result [**1**, p. 27] that for any *hh*-simple set A and any $\boldsymbol{d} \in \boldsymbol{H}_1$ there exists an r.e. $B \in \boldsymbol{d}$ such that $\mathscr{L}(A) \simeq \mathscr{L}(B)$.

In contrast to this "complexity" of sets of high degree, we might expect those of low degree to exhibit some "uniformity" of structure like that of recursive sets which fall into only three distinct $\mathscr{E}^*$-orbits. We cannot expect $A \equiv_{\mathscr{E}} B$ for all infinite, coinfinite $A, B \in \mathscr{E}$ of low degree because such sets may be recursive, simple, or neither. However, such sets A exhibit uniformity of $\mathscr{L}(A)$. For example, R. W. Robinson [**9**] verified Martin's conjecture that $\boldsymbol{A} \cap \boldsymbol{L}_1 = \varnothing$ by constructing a maximal superset B for any coinfinite $A \in \mathscr{E}$ of low degree. Lachlan [**1**, p. 27] showed that "B maximal" above could be replaced by "B *hh*-simple with $\mathscr{L}(B) \simeq \mathscr{L}(C)$" where C is an arbitrary *hh*-simple set. Lachlan then conjectured that $\mathscr{L}(A) \simeq \mathscr{L}(B)$ for any A, B both simple and of low degree.

For any set $A \subseteq N$ (not necessarily r.e.) define

$$\mathscr{E}_A = \{W \cap A : W \in \mathscr{E}\}.$$

If A is r.e. and infinite then $\mathscr{E}_A \simeq \mathscr{E}$ of course. The above results are corollaries of the following theorem, because if B is r.e. then clearly

$$\mathscr{E}_B \simeq \mathscr{L}(B).$$

THEOREM 3.3. *If A is infinite and* $\deg A \in \boldsymbol{L}_1$ *then $\mathscr{E}_A \simeq \mathscr{E}$.*

Carl Jockusch noted that our proof of Theorem 3.3 uses only the weaker hypothesis that $\{e : W_e \cap A \neq \varnothing\}$ has degree $\mathbf{0}'$. Such sets exist in every r.e. degree. Further discussion and open questions will appear at the end of [**15**].

ADDED IN PROOF (OCTOBER 22, 1973). The relativized lattice $\mathscr{E}_A$ may be used to give a structural characterization of degrees $\mathbf{a} \leqq \mathbf{0}'$ (not necessarily r.e.) which are high or low. Let $\mathscr{E}_A^* \cong \text{eff}_{\mathscr{E}^*}$ denote that there is a recursive permutation h of N such that the correspondence $(W_x \cap A)^* \leftrightarrow W_{h(x)}^*$, $x \in N$, gives an isomorphism from $\mathscr{E}_A^*$ to $\mathscr{E}^*$. Using Theorem 3.3 we can prove that for $\mathbf{a} \leqq \mathbf{0}'$,

$$\mathbf{a}' = \mathbf{0}' \Leftrightarrow (\forall \text{ infinite set } A \in \mathbf{a})[\mathscr{E}_A^* \cong {}^{\text{eff}}\mathscr{E}^*].$$

(This answers a question of Hay.) On the other hand, Morley and the author have extended Lachlan's theorem [**1**] by showing that for A infinite and in Δ_2^0, A is hyperhyperimmune if and only if $\mathscr{E}_A$ is a Boolean algebra. Combining this with results of Cooper and Jockusch we see that for $\mathbf{a} \cong \mathbf{0}'$,

$$\mathbf{a}' = \mathbf{0}'' \Leftrightarrow (\exists \text{ infinite set } A \in \mathbf{a})[\mathscr{E}_A \text{ is a Boolean algebra}].$$

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