AUTOMORPHISMS OF THE LATTICE OF RECURSIVELY ENUMERABLE SETS

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Let $\mathcal{E}$ denote the lattice of recursively enumerable (r.e.) sets under inclusion, and let $\mathcal{E}^*$ denote the quotient lattice of $\mathcal{E}$ modulo the ideal $\mathcal{F}$ of finite sets. For $A \in \mathcal{E}$ let $A^*$ denote the equivalence class in $\mathcal{E}^*$ which contains $A$. An r.e. set $A$ is maximal if $A^*$ is a coatom (maximal element) of $\mathcal{E}^*$. Let $\text{Aut } \mathcal{E}$ (Aut $\mathcal{E}^*$) denote the group of automorphisms of $\mathcal{E}$ ($\mathcal{E}^*$). We prove that, for any two maximal sets $A$ and $B$, there exists $\Phi \in \text{Aut } \mathcal{E}$ such that $\Phi(A) = B$. It follows that for each $k \geq 1$ the group Aut $\mathcal{E}^*$ is $k$-ply transitive on its coatoms. This demonstrates much more uniformity of structure of $\mathcal{E}$ than was supposed, and answers a question of Martin and Lachlan [1, p. 36]. We also use automorphisms to relate the structure of an r.e. set to its degree, particularly for degrees $d$ which are high ($d' = 0''$) or low ($d' = 0'$), and as corollaries we answer questions and extend results of Lachlan, Martin, Sacks, Yates, and others. The proofs involve infinite-injury priority arguments like those of Sacks [11], [12], and [13], but here an altogether different method is needed to resolve conflicts between opposing requirements. The numbering of results in §1 and §2 corresponds to that of [15] where full proofs will appear. The results in §3 will appear in [16] and [17].

1. Background information. For $A, B \in \mathcal{E}$, let $A \equiv_\mathcal{E} B$ ($A^* \equiv_\mathcal{E} B^*$) denote that there exists $\Phi \in \text{Aut } \mathcal{E}$ (Aut $\mathcal{E}^*$) such that $\Phi(A) = B$ ($\Phi(A^*) = B^*$). A permutation $p$ of $\mathcal{N}$ induces an automorphism $\Phi$ of $\mathcal{E}$ (or $\mathcal{E}^*$) if


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for all $W \in \mathcal{C}$, $\Phi(W) \equiv p(W)$ ($\Phi(W^*) = (p(W))^*$). Note that every recursive permutation induces some $\Phi \in \text{Aut } \mathcal{C}$, and every $\Phi \in \text{Aut } \mathcal{C}$ induces some $\Psi \in \text{Aut } \mathcal{C}$. The following results answer questions of Rogers [10, pp. 228–229].

**Theorem 1.1** (Lachlan). There are $2^{80}$ automorphisms of $\mathcal{C}^*$.

**Corollary 1.2.** Not every $\Phi \in \text{Aut}(\mathcal{C}^*)$ is induced by a recursive permutation of $\mathbb{N}$.

**Theorem 1.3.** Every $\Phi \in \text{Aut}(\mathcal{C}^*)$ is induced by some permutation of $\mathbb{N}$.

**Corollary 1.4.** Every $\Phi \in \text{Aut}(\mathcal{C}^*)$ is induced by some $\Psi \in \text{Aut}(\mathcal{C})$.

**Corollary 1.5.** If $A, B \in \mathcal{C}$ are infinite and co-Finite then $A \equiv_{\mathcal{C}} B$ if and only if $A^* \equiv_{\mathcal{C}} B^*$.

Thus, a property of r.e. sets which are well defined on $\mathcal{C}^*$ is invariant under $\text{Aut } \mathcal{C}^*$ just if it is invariant under $\text{Aut } \mathcal{C}$ in which case it is called invariant. By these corollaries from now on we can use $\mathcal{C}$ and $\mathcal{C}^*$ “interchangeably” selecting whichever is more convenient. In Lachlan’s method (Theorem 1.1) $\Phi \in \text{Aut } \mathcal{C}^*$ is induced by a permutation $p$ which is obtained by “piecing together” recursive permutations in a non-recursive fashion, but no new elements are produced in the $\mathcal{C}$-orbit of $A \in \mathcal{C}$, namely the equivalence class $\{B: A \equiv_{\mathcal{C}} B\}$. To do this requires a more effective construction such as that of Martin, which can be used to prove the noninvariance of hypersimplicity (Theorem 1.9), creativeness (Theorem 1.10), and Turing degree. Martin’s idea is to induce $\Phi \in \text{Aut } \mathcal{C}$ by a permutation which although not recursive is the limit of recursive permutations. To insure that $p$ induces an automorphism, one attempts to interchange elements only if they have the same “e-state”.

2. **Automorphisms and maximal sets.** Let $\{W_n\}_{n \in \mathbb{N}}$ be an acceptable numbering of the r.e. sets [10, p. 41]. We call $\Phi \in \text{Aut } \mathcal{C}^*$ effective if there is some recursive permutation $h$ of $\mathbb{N}$ such that $\Phi(W_n) = W_{h(n)}$ for all $n$. All $\Phi \in \text{Aut } \mathcal{C}^*$ produced by Martin’s “finite-injury” method (Theorem 1.9) are effective, roughly because one has such strong control over the inducing permutation $p$ and the two sets being constructed. Effective automorphisms, however, will not suffice for the maximal set case, as we discovered after Lachlan suggested that there might be no uniform way to prove $A \equiv_{\mathcal{C}} B$ given maximal sets $A$ and $B$.

**Theorem 2.1.** There exist maximal sets $A$ and $B$ such that $\Phi(A^*) \neq B^*$ for every effective $\Phi \in \text{Aut } \mathcal{C}^*$.
On the other hand our first principal result is that noneffective automorphisms will suffice.

**Theorem 2.3.** If \( A \) and \( B \) are maximal sets then \( A \equiv_\delta B \).

We call \( A \in \mathcal{E} \) quasimaximal of rank \( n \) if \( A \) is the intersection of \( n \) maximal sets \( M_i, 1 \leq i \leq n \), whose complements are pairwise disjoint. By applying Theorem 2.3 to each of \( n \) disjoint recursive sets which separate the \( (M_i)^-, 1 \leq i \leq n \), we see that \( A \equiv_\delta B \) if \( A \) and \( B \) are quasimaximal of the same rank (Corollary 2.6). Hence, for all \( k \geq 1 \), the group \( \text{Aut} \mathcal{E}^* \) is \( k \)-ply transitive on the coatoms of \( \mathcal{E}^* \) (Corollary 2.7).

A class \( \mathcal{C} \subseteq \mathcal{E} \) is a skeleton if \( \mathcal{C}^* = \mathcal{E}^* \) where \( \mathcal{C}^* = \{ W^*: W \in \mathcal{C} \} \). To prove Theorem 2.3 we first choose appropriate skeletons \( \{U_n\}_{n \in \mathbb{N}} \) and \( \{V_n\}_{n \in \mathbb{N}} \) depending upon \( A \) and \( B \), respectively. To specify the automorphism we give a permutation \( p \) of \( \mathbb{N} \) and recursive functions \( f \) and \( g \) such that for all \( n \), \( W^*_f(n) = (p(U_n))^* \) and \( W^*_g(n) = (p^{-1}(V_n))^* \). The heart of the method is a difficult result called the Extension Theorem (Theorem 2.2) which gives certain sufficient conditions under which a partial mapping on \( \mathcal{E} \) can be extended to an automorphism. The proof is presented in the informal style of Lerman's "pinball machines" [5] to give the reader a clear picture of the dynamics of the construction.

**3. The structure of an r.e. set and its degree.** Recursion theorists have been interested in the relationship between the structure of an r.e. set \( A \) and its degree, denoted by \( \text{deg} A \), ever since Post [8] asked for a simple property on the complement of \( A \in \mathcal{E} \) which guarantees incompleteness, i.e., \( 0 < \text{deg} A < 0' \). The existence of such a property remains an open question [11, Q(3), p. 172], although we give a partial answer by showing that no such property can be lattice-invariant, as are the properties of simplicity, hyper-hypersimplicity, and maximality.

**Theorem 3.1.** For any nonrecursive \( A \in \mathcal{E} \) there exists \( B \in \mathcal{E} \) of degree \( 0' \) such that \( A \equiv_\delta B \).

A corollary is Yates result [18] that there is a complete maximal set.

Let \( \mathcal{R} \) denote the class of r.e. degrees. For each \( n \geq 0 \) define the subclasses of \( \mathcal{R} \),

\[
H_n = \{ d: d \text{ r.e. and } d^{(n)} = 0^{(n+1)} \}, \quad \text{and}
\]

\[
L_n = \{ d: d \text{ r.e. and } d^{(n)} = 0^{(n)} \},
\]

where \( d^{(0)} = d \). It is well known [10, pp. 290–294] that for each \( n, H_n \subseteq H_{n+1} \) and \( L_n \subseteq L_{n+1} \) and that there exists an r.e. degree \( d \) such that for all \( n, d \notin H_n \cup L_n \). The degrees in \( H_1 (L_1) \) are called high (low). (This terminology
is often used with the condition “d r.e.” above replaced by the weaker condition “d ≤ 0”.

A class \( C \subseteq \mathcal{R} \) is called \( \mathcal{E} \)-definable if \( C = \{ \deg W : W \in \mathcal{E} \} \) for some lattice-invariant class \( \mathcal{E} \subseteq \mathcal{E} \). For example, let \( \mathcal{M} \) denote the class of degrees of maximal sets, and \( \mathcal{A} \) the class of degrees of atomless sets, that is coinfinite r.e. sets which have no maximal supersets. The beautiful theorems of Martin [7] assert that \( \mathcal{M} = H_1 \) and \( \mathcal{A} \supseteq H_1 \). The difficult parts of Martin’s results, namely that \( \mathcal{M} \supseteq H_1 \) and \( \mathcal{A} \supseteq H_1 \), can be obtained from the following result [16] suggested by Carl Jockusch. For \( A \in \mathcal{E} \) define \( \mathcal{L}(A) = \{ W : W \in \mathcal{E} \wedge W \supseteq A \} \). Note that \( \mathcal{L}(A) \) is a lattice under inclusion.

**Theorem 3.2.** For any nonrecursive \( A \in \mathcal{E} \) and any \( d \in H_1 \) there exists an r.e. \( B \in d \) such that \( \mathcal{L}(A) \cong \mathcal{L}(B) \).

Thus, no isomorphism-invariant property on \( \mathcal{L}(A) \) can guarantee either completeness or incompleteness. Another corollary is Lachlan’s result [1, p. 27] that for any hh-simple set \( A \) and any \( d \in H_1 \) there exists an r.e. \( B \in d \) such that \( \mathcal{L}(A) \cong \mathcal{L}(B) \).

In contrast to this “complexity” of sets of high degree, we might expect those of low degree to exhibit some “uniformity” of structure like that of recursive sets which fall into only three distinct \( \mathcal{E}^* \)-orbits. We cannot expect \( A \equiv_\mathcal{E} B \) for all infinite, coinfinite \( A, B \in \mathcal{E} \) of low degree because such sets may be recursive, simple, or neither. However, such sets \( A \) exhibit uniformity of \( \mathcal{L}(A) \). For example, R. W. Robinson [9] verified Martin’s conjecture that \( A \cap L_1 = \emptyset \) by constructing a maximal superset \( B \) for any coinfinite \( A \in \mathcal{E} \) of low degree. Lachlan [1, p. 27] showed that “\( B \) maximal” above could be replaced by “\( B \) hh-simple with \( \mathcal{L}(B) \cong \mathcal{L}(C) \)” where \( C \) is an arbitrary hh-simple set. Lachlan then conjectured that \( \mathcal{L}(A) \cong \mathcal{L}(B) \) for any \( A, B \) both simple and of low degree.

For any set \( A \subseteq \mathcal{N} \) (not necessarily r.e.) define

\[
\mathcal{E}_A = \{ W \cap A : W \in \mathcal{E} \}.
\]

If \( A \) is r.e. and infinite then \( \mathcal{E}_A \cong \mathcal{E} \) of course. The above results are corollaries of the following theorem, because if \( B \) is r.e. then clearly

\[
\mathcal{E}_B \cong \mathcal{L}(B).
\]

**Theorem 3.3.** If \( A \) is infinite and \( \deg A \in L_1 \) then \( \mathcal{E}_A \cong \mathcal{E} \).

Carl Jockusch noted that our proof of Theorem 3.3 uses only the weaker hypothesis that \( \{ e : W_e \cap A \neq \emptyset \} \) has degree \( 0' \). Such sets exist in every r.e. degree. Further discussion and open questions will appear at the end of [15].

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ADDED IN PROOF (OCTOBER 22, 1973). The relativized lattice $\mathcal{E}_A$ may be used to give a structural characterization of degrees $a \leq 0'$ (not necessarily r.e.) which are high or low. Let $\mathcal{E}_A^* \equiv_{\text{eff}} \mathcal{E}_A^*$ denote that there is a recursive permutation $h$ of $\mathbb{N}$ such that the correspondence $(W_x \cap A)^* \leftrightarrow W_x^{(h(x))}$, $x \in \mathbb{N}$, gives an isomorphism from $\mathcal{E}_A^*$ to $\mathcal{E}_A^*$. Using Theorem 3.3 we can prove that for $a \leq 0'$,

$$a' = 0' \iff (\forall \text{ infinite set } A \in a)[\mathcal{E}_A^* \equiv_{\text{eff}} \mathcal{E}_A^*].$$

(This answers a question of Hay.) On the other hand, Morley and the author have extended Lachlan’s theorem [1] by showing that for $A$ infinite and in $\Delta_2^e$, $A$ is hyperhyperimmune if and only if $\mathcal{E}_A$ is a Boolean algebra. Combining this with results of Cooper and Jockusch we see that for $a \equiv 0'$,

$$a' = 0'' \iff (\exists \text{ infinite set } A \in a)[\mathcal{E}_A^* \text{ is a Boolean algebra}].$$

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