COMPLETION AND EMBEDDING BETWEEN PSEUDO 
(v, k, λ)-DESIGNS AND (v, k, λ)-DESIGNS

BY OSVALDO MARRERO

Communicated by Dock Rim, May 29, 1973

ABSTRACT. Each of four arithmetical conditions on the parameters v, k, and λ of a given primary pseudo (v, k, λ)-design is necessary and sufficient to ensure completion or embedding between the given design and some (v', k', λ')-design.

Let \( X = \{x_1, \ldots, x_v\} \), and let \( X_1, \ldots, X_v \) be subsets of \( X \). The subsets \( X_1, \ldots, X_v \) are said to form a \((v, k, \lambda)\)-design if

1. each \( X_j \) (\( 1 \leq j \leq v \)) has \( k \) elements;
2. any two distinct \( X_i, X_j \) (\( 1 \leq i, j \leq v \)) intersect in \( \lambda \) elements; and
3. \( 0 \leq \lambda < k < v - 1 \).

Such a design is completely determined by its incidence matrix; this is the \((0, 1)\)-matrix \( A = [a_{ij}] \) defined by taking \( a_{ij} = 1 \) if \( x_j \in X_i \) and \( a_{ij} = 0 \) if \( x_j \notin X_i \). More information about these combinatorial designs is available, for example, in [2] and [5].

Let \( Y = \{y_1, \ldots, y_v\} \), and let \( Y_1, \ldots, Y_{v-1} \) be subsets of \( Y \). The subsets \( Y_1, \ldots, Y_{v-1} \) are said to form a pseudo \((v, k, \lambda)\)-design if

1. each \( Y_j \) (\( 1 \leq j \leq v-1 \)) has \( k \) elements;
2. any two distinct \( Y_i, Y_j \) (\( 1 \leq i, j \leq v-1 \)) intersect in \( \lambda \) elements; and
3. \( 0 < \lambda < k < v - 1 \).

The incidence matrix of a pseudo \((v, k, \lambda)\)-design is defined in the same manner as the incidence matrix of a \((v, k, \lambda)\)-design.

The consideration of pseudo \((v, k, \lambda)\)-designs was suggested during the course of study of “modular hadamard matrices” [3], [4]. Related work has been published by Bridges [1] and Woodall [6].

A pseudo \((v, k, \lambda)\)-design is “almost” (its incidence matrix lacks one row) a \((v, k, \lambda)\)-design; this suggests the consideration of “completion and embedding” between these two combinatorial designs. Let \( A \) be the incidence matrix of a pseudo \((v, k, \lambda)\)-design. Then it might be possible to

---


Key words and phrases. Block designs, \((v, k, \lambda)\)-designs, pseudo \((v, k, \lambda)\)-designs, completion and embedding of block designs.

Copyright © American Mathematical Society 1974

103
"complete" the \( v - 1 \) rows of \( A \) by adjoining one additional row to \( A \), and possibly performing some operations on the rows or columns of \( A \), so that the incidence matrix of some \((v, k', \lambda')\)-design is obtained; also, it might be possible that the incidence matrix of some \((v-1, k', \lambda')\)-design is "embedded" in \( A \). This paper presents a theorem and a conjecture dealing with such completion and embedding. No proof of the theorem below is given in this paper. A more comprehensive paper dealing with pseudo \((v, k, \lambda)\)-designs is being planned by this author.

When \( k=2\lambda \), the existence of a pseudo \((v, k, \lambda)\)-design implies and is implied by the existence of some \((v', k', \lambda')\)-design; and, if the parameters of a given pseudo \((v, k, \lambda)\)-design satisfy \( v\lambda=k^2 \), then they must satisfy \( k=2\lambda \) [3]. A pseudo \((v, k, \lambda)\)-design is called primary or nonprimary according to whether its parameters satisfy \( v\lambda\neq k^2 \) or \( v\lambda=k^2 \), respectively. Thus, it is the existence of primary pseudo \((v, k, \lambda)\)-designs which remains unresolved.

The incidence matrix of a pseudo \((v, k, \lambda)\)-design can be obtained from the incidence matrix \( A \) of a given \((v', k', \lambda')\)-design by any one of the following four simple techniques:

1. a column of \(+1\)'s is adjoined to \( A \);
2. a column of \(0\)'s is adjoined to \( A \);
3. a row is discarded from \( A \); or
4. a row is discarded from \( A \) and then the \( k' \) columns of \( A \) which had a \(+1\) in the discarded row are complemented (\(0\)'s and \(+1\)'s are interchanged in these columns).

These four are the only known techniques for the construction of pseudo \((v, k, \lambda)\)-designs. The initial observation which led to the theorem below is that there is a simple arithmetical condition on the parameters \( v, k, \) and \( \lambda \) which is necessary for the incidence matrix of a given primary pseudo \((v, k, \lambda)\)-design to be obtained from the incidence matrix of some \((v', k', \lambda')\)-design by one of the aforementioned techniques; it can be shown that each one of these conditions is also sufficient, thus answering the completion and embedding problem under consideration in these four cases.

**Theorem.** The incidence matrix of a given primary pseudo \((v, k, \lambda)\)-design can be obtained from the incidence matrix of some \((v', k', \lambda')\)-design by the \( i \)-th \((1 \leq i \leq 4)\) technique above if and only if the parameters \( v, k, \) and \( \lambda \) satisfy the respective \( i \)-th condition below:

1. \((k-1)(k-2)=(\lambda-1)(v-2)\);
2. \(k(k-1)=\lambda(v-2)\);
3. \(k(k-1)=\lambda(v-1)\); or
4. \(k=2\lambda\).
A primary pseudo \((v, k, \lambda)\)-design is said to be of type \(i\) \((1 \leq i \leq 4)\) if its parameters satisfy the \(i\)th equation in the statement of the above theorem. There are examples of pseudo \((v, k, \lambda)\)-designs of each of these four types that are not of any of the other three types. It is possible for a pseudo \((v, k, \lambda)\)-design to be of more than one type.

The condition that the parameters \(v, k, \lambda\) satisfy the \(i\)th \((1 \leq i \leq 4)\) equation in the statement of the theorem above is not sufficient to ensure the existence of a pseudo \((v, k, \lambda)\)-design, since none of these conditions is sufficient to ensure the existence of a \((v', k', \lambda')\)-design with the appropriate parameters \(v', k', \lambda'\).

This author has conjectured that given a primary pseudo \((v, k, \lambda)\)-design, then completion or embedding between the given design and some \((v', k', \lambda')\)-design must always be possible. The precise statement is:

**Conjecture.** The parameters of a given primary pseudo \((v, k, \lambda)\)-design must satisfy at least one of the equations in the statement of the above theorem.

It is known that the above conjecture is valid whenever \(\lambda = 1\).

**References**


**Department of Mathematics, Francis Marion College, Florence, South Carolina 29501**