ON HILBERT TRANSFORMS ALONG CURVES

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Let $\gamma(t), -\infty < t < \infty$, be a smooth curve in $\mathbb{R}^n$. For $f$ in $C_0^\infty(\mathbb{R}^n)$ set

\begin{equation}
Tf(x) = \lim_{\varepsilon \to \infty, N \to \infty} \int_{\varepsilon \leq |t| \leq N} \frac{f(x - \gamma(t))}{t} \, dt.
\end{equation}

$Tf$ is the Hilbert transform of $f$ along the curve $\gamma(t)$. E. M. Stein [2] raised the following general question: For what values of $p$ and what curves $\gamma(t)$ is $Tf$ a bounded operator in $L^p$? If $\gamma(t)$ is a straight line it is well known that $T$ is bounded for $1 < p < \infty$. Stein and Wainger [3] proved that the operator is bounded for $p = 2$ if

\[ \gamma(t) = (|t|^\alpha \operatorname{sgn} t, \cdots, |t|^\alpha \operatorname{sgn} t), \quad \alpha > 0. \]

Here we show that $Tf$ is a bounded operator in $L^p$ for some $p$ other than 2 and some nontrivial, nonlinear $\gamma$'s. We prove

**Theorem 1.** Let $\gamma(t) = (|t|^{\alpha_1} \operatorname{sgn} t, |t|^{\alpha_2} \operatorname{sgn} t)\alpha_1 > 0, \quad \alpha_2 > 0$. Then $Tf$ is bounded in $L^p$ for $\frac{2}{3} < p < 4$.

**Sketch of the Proof.** The transformation (1) may be expressed as a multiplier transformation. In our case,

\begin{equation}
(Tf)^\wedge(x, y) = m(x, y)\hat{f}(x, y)
\end{equation}

where

\begin{equation}
m(x, y) = \lim_{\varepsilon \to \infty, N \to \infty} \int_{\varepsilon \leq |t| \leq N} \exp\{i |t|^{\alpha_1} \operatorname{sgn} tx + i |t|^{\alpha_2} \operatorname{sgn} ty\} \frac{dt}{t}
\end{equation}

($\hat{}$ denotes Fourier transform).

By a change of variables we may assume $\alpha_1 = 1$ and $\alpha_2 \geq 1$. Furthermore we may assume $\alpha_2 > 1$, for otherwise we have the case that $\gamma(t)$ is a straight
line. Thus in (3) we take \( \alpha_1 = 1 \) and \( \alpha_2 = \alpha > 1 \). Clearly \( m \) is odd and \( m(rx, r^2y) = m(x, y), r > 0 \). By using the method of steepest descents and integration by parts we obtain

**Theorem 2.** \( m(x, y) \) is infinitely differentiable away from the line \( y = 0 \).

For \( 0 \leq |y|/x^* \leq 1 \),

\[
m(x, y) = m_1(x, y) + m_2(x, y) + m_3(x, y),
\]

where, if

\[
\lambda = |y| x^-^a \quad \text{and} \quad \beta = (a - 1)^{-1}
\]

\[
m_1(x, y) = \sum_{j=1}^{n} A_j \lambda^{\beta / 2 + n_j} \exp(i \lambda^{-\beta} \eta_j), \quad y \geq 0,
\]

\[
m_2(x, y) = \sum_{j=1}^{m} B_j \lambda^{\beta / 2 + \rho_j} \exp(i \lambda^{-\beta} \xi_j), \quad y \leq 0,
\]

\[
m_3(x, y) = \sum_{j=1}^{m} B_j \lambda^{\beta / 2 + \rho_j} \exp(i \lambda^{-\beta} \xi_j), \quad y \geq 0,
\]

\( m_3(x, y) \) has continuous second order partial derivatives away from the origin. Here \( A_j \) and \( B_j \) are complex numbers \( \eta_j \geq 0, \rho_j \geq 0, \) and \( \nu_j \) and \( \xi_j \) are real.

We shall consider a multiplier of the form \( n(x, y) = g(y/x^*) \) where

\[
g(\lambda) = \begin{cases} 
\lambda^{\beta / 2} \exp(i \lambda^{-\beta}) \omega(\lambda), & \lambda > 0, \\
0, & \lambda \leq 0,
\end{cases}
\]

where \( \omega \) is \( C^\infty \), has support in \([-1, 1]\) and is identically 1 near \( \lambda = 0 \). Theorem 2 implies that \( m(x, y) \) is a finite sum of multipliers each of which may be treated in the same way as \( n(x, y) \). Set

\[
g_\sigma(\lambda) = \begin{cases} 
\lambda^{\sigma \beta} \exp(i \lambda^{-\beta}) \omega(\lambda), & \lambda \geq 0, \\
0, & \lambda \leq 0,
\end{cases}
\]

and \( n_\sigma(x, y) = g_\sigma(y/x^*) \).

We wish to show

\( n_{1/2} \) is a bounded multiplier on \( L^p \) for \( \frac{4}{3} < p < 4 \).

Clearly \( n_{\sigma+t(t)}(x, y) \) is a bounded multiplier on \( L^2 \) (with norm uniformly bounded in \( \sigma \)). Hence, in view of the interpolation theorem for analytic families of operators, to prove \( n_{1/2} \) is a bounded multiplier on \( L^p, \frac{4}{3} < p < 4 \), it suffices to prove

**Theorem 3.** \( n_{\sigma+t(t)} \) is a bounded multiplier on \( L^p, 1 < p < \infty \) for \( \sigma > 1 \), with a bound that is independent of \( t \).
Theorem 3 will in turn follow by arguments similar to Rivière [1], if one can prove the following

**Lemma.** Let $\psi(r)$ be in $C^\infty[0, \infty)$ with support in $[\frac{1}{2}, 2]$, $\rho(x, y) = (x^2 + y^2)^{1/2}$, and $\phi(x, y) = \psi(\rho(x, y))$. For $\delta$ positive and small set $l = \frac{1}{2}(1 + 1/\delta) + \delta$ and $k = (x + 1)/2$.

Then

\[ \int_{|x|^2 + |y|^2} (n_{\sigma+it})^2 (n_{\sigma+it}) \psi(x, y) \, dx \, dy \leq C \]

\[ \int_{|x|^2 + |y|^2} (n_{\sigma+it})^2 (n_{\sigma+it}) \psi(x, y) \, dx \, dy \leq C[\rho(s, u)]^2. \]

$h_{s,u}(x, y) = e^{i(xs + yu)} - 1$. ($\psi$ denotes inverse Fourier transform).

Lemma 2 is proved by (a) proving appropriate analogues of (i) and (ii) if $k = m + it$, $m$ a nonnegative integer, $l = 1 + it$, and $l = it$, and then (b) using the Phragmén-Lindelöf theorems. Details will appear elsewhere.

**References**


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