QUASI-KAN EXTENSIONS FOR 2-CATEGORIES

BY JOHN W. GRAY

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1. Introduction. Let \( \text{Cat} \) denote the category of small categories and functors. \( \text{Cat} \) is a Cartesian closed category, [2] and the prefix \( 2- \) will denote categories and functors enriched in \( \text{Cat} \). \( 2-\text{Cat} \) denotes the category of small 2-categories and 2-functors. It is also Cartesian closed, but there is another notion of a transformation between 2-functors \( F \) and \( G \) which has interesting properties; namely a quasi-natural transformation from \( F \) to \( G \) is a family of morphisms \( \{ \varphi_A : F(A) \rightarrow G(A) \} \) together with a family of 2-cells \( \{ \varphi_f : G(f) \varphi_A \rightarrow \varphi_B F(f) \} \) as illustrated

\[
\begin{array}{ccc}
F(A) & \xrightarrow{F(f)} & F(B) \\
\downarrow \varphi_A & & \downarrow \varphi_B \\
G(A) & \xrightarrow{G(f)} & G(B)
\end{array}
\]

satisfying obvious compatibility conditions. (The case where the \( \varphi_f \)'s are isomorphisms has been considered in [7] and [8], but we make no such restriction.) Given this notion of "natural transformation", it is reasonable and useful to inquire about the corresponding notion of "quasi-limit" or, more generally, "quasi-Kan extension".

Such a Kan extension was used in an essential way for the proof of the main result in [4, §9], but until now no justification has been given for calling the construction used there a "Kan extension". In the usual case, if \( S : \mathcal{A} \rightarrow \mathcal{B} \) is an ordinary functor and \( \mathcal{X} \) is a cocomplete category, then under appropriate hypotheses the functor

\( \mathcal{X}^S : \mathcal{X}^B \rightarrow \mathcal{X}^A \)

is right adjoint to the (left) Kan extension \( \Sigma S : \mathcal{X}^A \rightarrow \mathcal{X}^B \). \( \Sigma S \) can be constructed as follows: replace \( S \) by its associated factorization through an opfibration

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{Q_S} & (S, \mathcal{B}) \\
\downarrow P & & \downarrow P_S \\
\mathcal{A} & \rightarrow & \mathcal{B}
\end{array}
\]
where $S=P\circ Q$ and $P$ is left inverse, right adjoint to $Q$ (see [3, p. 55]). One shows that for the opfibration $P$, the Kan extension $\Sigma P$ is given by "integration (i.e., colimit) along the fibres" and then that $\Sigma S=(\Sigma P)\times^{\mathcal{E}}$.

In the 2-category case, 2-functors and quasi-natural transformations are the objects and morphisms of a 2-category $\text{Fun}(\mathcal{A}, \mathcal{B})$ which is the internal hom object for a nonsymmetrical, monoidal closed structure on $2\text{-}\text{Cat}$, denoted by $2\text{-}\text{Cat}_\otimes$. (Cf. [4, p. 280] and [6].) If $S: \mathcal{A} \to \mathcal{B}$ is a fixed 2-functor, then for any $\mathcal{X}$, there is an induced functor

$$S^* = \text{Fun}(S, \mathcal{X}): \text{Fun}(\mathcal{B}, \mathcal{X}) \to \text{Fun}(\mathcal{A}, \mathcal{X})$$

and we can ask for a left quasi-adjoint $\Sigma S$ to $S^*$ in some suitable sense of quasi-adjointness. (The case $\mathcal{B}=1$ yields quasi-colimits.) In this paper we describe how to modify each step in the procedure described above to fit the situation of 2-categories. It will be seen that when $S$ is a suitable kind of quasi-opfibration, one obtains an ordinary $\mathcal{E}$-enriched adjoint ($=\text{Cat}$-adjoint). This includes the case of quasi-limits. However, in general one gets a strict quasi-adjoint. This notion forces itself upon one when one studies $2\text{-}\text{Cat}_\otimes$ seriously, since it arises in many different contexts. (In particular, the comprehension scheme in [4] is a strict quasi-adjoint, as are all the constructions mentioned in this paper.) Detailed proofs will be published in [6].

2. Definitions. Besides the 2-comma category $[S_1, S_2]$ defined as in [4, p. 279], for a pair of 2-functors $S_1: \mathcal{A} \to \mathcal{B}$, there are 3-comma categories $[S_1, S_2]_3$ and $[S_1, S_2]_\otimes$ defined for 3-functors (resp. 2-category-functors) $S_1$ and $S_2$ between 3-categories (resp. 2-category-categories) with the same codomain. 0-cells and 1-cells are defined as in $[S_1, S_2]$, 2-cells are a pair of 2-cells as in $[S_1, S_2]$ plus a 3-cell expressing the lack of commutativity, and 3-cells are pairs satisfying the obvious equation. Details will be given in [6].

Let $F: \mathcal{A} \to \mathcal{B}$ and $U: \mathcal{B} \to \mathcal{A}$ be 2-functors between 2-categories. A pair of quasi-natural transformations, $\varepsilon: FU \to \mathcal{B}$ and $\eta: \mathcal{A} \to UF$ is called a quasi-adjunction between $F$ and $U$ if it satisfies the usual equations. It is called strict if

$$(U\varepsilon F)(\eta_\eta) = 1_\eta, \quad \varepsilon_\varepsilon(F\eta U) = 1_\varepsilon.$$

Here, for instance, since $\eta_\mathcal{A}: \mathcal{A} \to UF\mathcal{A}$ is an arrow, $\varepsilon$ assigns to it a 2-cell from $(UF\eta_\mathcal{A})\eta_\mathcal{A}$ to $(\eta UF\mathcal{A})\eta_\mathcal{A}$. This defines the modification (2-cell in $\text{Fun}(\mathcal{A}, \mathcal{A})$) denoted by $\eta_\varepsilon$. Similarly, $1_\eta$ is the identity modification of $\eta$. 

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3. Cartesian quasi-limits and colimits. In the special case of induced functors $S^*$ in which $\mathcal{B} = \mathbf{1}$, one obtains the constant embedding
\[ \Delta_X : \mathcal{X} \rightarrow \text{Fun}(\mathcal{A}, \mathcal{X}). \]
The right (resp. left) $\mathcal{C}$-adjoint to $\Delta_X$ is called the Cartesian quasi-limit (resp., colimit) of type $\mathcal{A}$ in $\mathcal{X}$ and is denoted by
\[ Q_{\mathcal{C}} \quad \Delta_{\mathcal{X}} \quad \Delta_{\mathcal{C}} \quad Q_{\mathcal{A}} \]
$\mathcal{X}$ is called Cartesian quasi-complete (resp., cocomplete) if $Q_{\mathcal{C}}$ (resp., $Q_{\mathcal{A}}$) exists in $\mathcal{X}$ for all small 2-categories $\mathcal{A}$.

**Theorem 1.** $\mathcal{C}al$ is Cartesian quasi-complete and cocomplete.

**Proof.** We describe the construction here.

Let $H : \mathcal{A} \rightarrow \mathcal{C}al$ be a 2-functor. $H$ determines an opfibration $P : \mathcal{E}_H \rightarrow \mathcal{A}$ (see [4, pp. 248, 285, 289]) where $\mathcal{E}_H$ is a 2-category. In the special case that $H : \mathcal{A} \rightarrow \mathcal{E}_\mathcal{A}$ with $\mathcal{A}$ an ordinary category, then $\lim H = \pi_0(\mathcal{E}_H)$ and $\lim H = \Gamma(\mathcal{E}_H)$ where $\pi_0$ assigns to a category its set of connected components, and $\Gamma$ is the set of sections of $P$. In the general case one shows that $Q_H = L\pi_0(\mathcal{E}_H)$ and $Q_H = \Gamma(\mathcal{E}_H)$ where $L\pi_0 : 2-\mathcal{C}al - \rightarrow \mathcal{C}al$ is “local $\pi_0$”, i.e., it turns a 2-category $\mathcal{C}$ into a category $L\pi_0(\mathcal{C})$ by replacing each hom-category $\mathcal{C}(X, Y)$ by the set $\pi_0(\mathcal{C})(X, Y)$. (Note that this differs from the assertion in [4, p. 289].) Similarly, $\Gamma$ denotes the category of sections of $P$; i.e., 2-functors $G : \mathcal{A} \rightarrow \mathcal{E}_H$ such that $PG = 1$ and natural transformations $\psi$ such that $P\psi = 1$.

In [5] it is asserted, and it will be proved elsewhere, that this result holds for strongly representable (resp., corepresentable) 2-categories. These are essentially 2-categories which are complete (resp., cocomplete) in the sense of closed categories.

We list here a number of examples of Cartesian quasi-limits and colimits in $\mathcal{C}al$. These examples serve to define the corresponding concepts in other 2-categories.

(i) $\mathcal{A} = 2$ (the category with two objects and a single nonidentity morphism). $H : 2 \rightarrow \mathcal{C}al$ looks like a functor $f : A \rightarrow B$ between small categories and $Q_H = (f, B)$, the universal opfibration associated to $f$, while $Q_H = (A, f)$, the universal cofibration associated to $f$ (see [3, §5]).

(ii) (cf. Street [10]) Let $\Delta^{op}$ denote the 2-category with a single object $\ast$, with $\text{Hom}(\ast, \ast)$ the dual of the category of finite ordinals. A 2-functor $H : \Delta^{op} \rightarrow \mathcal{C}al$ is the same as a small category $\mathcal{A}$ equipped with a cotriple $G$, and $Q_H$ is the co-Kleisli category of the cotriple. Let $\text{op}(\Delta^{op})$ be the weak dual. Then $H : \text{op}(\Delta^{op}) \rightarrow \mathcal{C}al$ is a small category equipped with a triple and $Q_H$ is the category of Eilenberg-Moore algebras. Appropriate duals give the other two possibilities.
If the possibilities are extended by allowing nonfull subcategories of $\text{Fun}(\mathcal{A}, \mathcal{B})$ determined by imposing conditions on the 2-cells $\varphi_i$ for certain $f$'s, then $\text{Cat}$ still admits such quasi-limits and colimits. As particular examples, one obtains comma categories [3] and subequalizers (Lambek [9]) as well as the result that the closure of $\mathcal{P} e l s \subset \text{Cat}$ under such quasi-colimits is all of $\text{Cat}$.

The main result about $Q$ needed for quasi-Kan extensions is the following.

**Theorem 2.** $Q: [\mathcal{E}, \text{Cat}_\otimes] \to \mathcal{X}$ is a $2\text{-Cat}_\otimes$-functor which is the left $2\text{-Cat}_\otimes$-adjoint to $N$.

Here $N$ is the functor in the other direction which is the name functor; e.g., on an object $X \in \mathcal{X}$, $N(X) = X: \mathcal{E} \to \mathcal{X}$, etc., and $s$ means small. The main (and considerable) difficulty is to show that $Q$ is defined here.

### 4. Quasi-fibrations.

Among the various possible definitions the following is the one needed here. Let $\text{Fun}(\mathcal{B})^{\text{op}} = \text{Fun}(2, \mathcal{B})^{\text{op}}$. A 2-functor $P: \mathcal{E} \to \mathcal{B}$ is called a Cartesian quasi-opfibration if there exists a 2-functor $L$ as illustrated

```
\begin{array}{ccc}
\text{Fun}(\mathcal{E}) & \xleftarrow{\delta_0} & [P, \mathcal{B}] \\
\downarrow{L} & & \downarrow{P} \\
\text{Fun}(\mathcal{P}) & \xleftarrow{\delta_0} & \text{Fun}(\mathcal{B})
\end{array}
```

having $R$ as a right $\text{Cat}$-adjoint and $RL = \text{id}$. Here the square is a pullback and $\delta_0$, $\text{Fun}(P)$ and $R$ are the obvious induced 2 functors. A choice of $L$ is called a cleavage. If $L$ is chosen so that $L(\text{id}) = \text{id}$ and

$$L(f \ast E, g) \circ L(E, f) = L(E, gf)$$

then $P$ together with $L$ is called a split-normal Cartesian quasi-opfibration. The 2-category of such together with cleavage preserving 2-functors and $\text{Cat}$ natural transformations over $\mathcal{B}$ is denoted by Cart $q$-Split($\mathcal{B}$)$_0$.

**Theorem 3.** The inclusion

$$\text{Cart } q\text{-Split}(\mathcal{B})_0 \to [\text{op}2 \text{Cat}, \mathcal{B}]$$

has a strict left quasi-adjoint $\Phi$.

Here $\Phi$ on an object $S: \mathcal{A} \to \mathcal{B}$ is the projection $P_S: [S, \mathcal{B}] \to \mathcal{B}$. 

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THEOREM 4. The associated opfibration has the property that there is a factorization of $S$,

$$
\mathcal{A} \xrightarrow{Q_S} [S, \mathcal{B}] \xrightarrow{P_S} \mathcal{B}
$$

in which $P_SQ_S = S$ and $P$ is left inverse, strict quasi-right adjoint to $Q_S$.

REMARK. $P_S$ is also an ordinary $\text{Cat}$-enriched opfibration and $P$ is a $\text{Cat}$-enriched fibration.

THEOREM 5. Let $(P: \mathcal{E} \to \mathcal{B}, L)$ be a split normal Cartesian quasi-opfibration with small fibres. Then there is a $2\text{-Cat}_\otimes$-imbedding $J: \mathcal{B} \to [2\text{-Cat}_\otimes, \mathcal{E}]_\otimes$.

5. Quasi-Kan extensions. For quasi-opfibrations, one has the following astonishing result.

THEOREM 6. If $P: \mathcal{E} \to \mathcal{B}$ is a split normal Cartesian quasi-opfibration with small fibres, and $\mathcal{X}$ is Cartesian quasi-cocomplete then $P^*: \text{Fun}(\mathcal{B}, \mathcal{X}) \to \text{Fun}(\mathcal{E}, \mathcal{X})$ has a left $\text{Cat}$-adjoint, $\Sigma_0P$ given by "integration along the fibres."

This means that if $G: \mathcal{E} \to \mathcal{X}$ is a 2-functor, then $\Sigma_0P(G)$ is the composition

$$
\mathcal{B} \xrightarrow{J} [2\text{-Cat}_\otimes, \mathcal{E}]_\otimes \xrightarrow{G^*} [2\text{-Cat}_\otimes, \mathcal{X}]_\otimes \xrightarrow{Q} \mathcal{X}
$$

where $G^*$ denotes composition with $G$.

Finally, we get the desired generalization of Kan extensions.

THEOREM 7. Let $S: \mathcal{A} \to \mathcal{B}$ be a 2-functor between small 2-categories and let $\mathcal{X}$ be Cartesian quasi-cocomplete. Let $\Sigma_0S = (\Sigma_0P_S)P^*$ where $P_S$ and $P$ are as in Theorem 4. Then $\text{op}(\Sigma_0S)$ is a strict quasi-left-adjoint to $\text{op}(S^*)$: $\text{op}\text{-Fun}(\mathcal{B}, \mathcal{X}) \to \text{op}\text{-Fun}(\mathcal{A}, \mathcal{X})$.

The claim in [4, §9], about $\Sigma_0S$ is incorrect and the adjunction is in the sense stated here. Part of the reason for the failure of $\Sigma_0S$ to be a $\text{Cat}$-adjoint in general is that $P$ is transversal to the fibres in $[S, \mathcal{B}]$ which has the effect that $\Sigma_0S$ applied to a quasi-natural transformation yields a $\text{Cat}$-natural transformation. An example of $\Sigma_0S$ is given in [4]. Others will be given elsewhere in the subject of 2-theories.

REFERENCES


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN, URBANA, ILLINOIS 61801