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ON MEASURABILITY, POINTWISE CONVERGENCE AND COMPACTNESS[1]

BY A. IONESCU TULCEA

The starting point of this investigation is the beautiful generalization of Egorov's theorem given by P. A. Meyer in Séminaire de Probabilités. V, (Strasbourg). The material is divided as follows:

§1. Setting and terminology.
§2. The Generalized Egorov Theorem.
§3. An application to vector-valued mappings.
§4. The ⟨⟨separation property⟩⟩ and the notion of lifting.

Proofs of most of the results contained in this paper can be found in [**5**], [**6**], [**7**], [**8**].

1. **Setting and terminology.** Throughout this article $(E, \mathscr{E}, \mu)$ will be a fixed probability space. We denote by $\mathscr{L}=\mathscr{L}(E, \mathscr{E}, \mu)$ the algebra of all $f: E\to R$ which are $\mathscr{E}$-measurable.

For $f \in \mathscr{L}$, $g \in \mathscr{L}$ we write

$$f \equiv g \quad \text{if } f(t) = g(t) \quad \mu\text{-almost surely},$$

and

$$f = g \quad \text{if } f(t) = g(t) \quad \text{for all } t \in E.$$

For $f \in \mathscr{L}$, we denote by $\tilde{f}$ the equivalence class of f with respect to the equivalence relation "$\equiv$" defined above.

We denote by $\mathscr{L}^\infty=\mathscr{L}^\infty(E, \mathscr{E}, \mu)$ the algebra of all *bounded* $f \in \mathscr{L}$.

For a set $B \in \mathscr{E}$ we denote by 1_B the indicator function of B (i.e. $1_B(t)=1$ for $t \in B$ and $1_B(t)=0$ for $t \in E-B$).

We say that a set $A \in \mathscr{E}$ *carries* μ if $\mu(E-A)=0$.

2. **The Generalized Egorov Theorem.** We may now state P. A. Meyer's generalization of Egorov's theorem (see [**8**, p. 199]) as follows:

THEOREM 1 (GENERALIZED EGOROV THEOREM). *Let $H \subset \mathscr{L}$ be compact and metrizable for the topology of pointwise convergence on E. There exists*

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then a sequence (A_n) of disjoint subsets of E such that $A_n \in \mathscr{E}$ for each n, and $\bigcup_n A_n$ carries μ, with the following property:

If (h_j) is any sequence of elements of H, converging pointwise on E, say to h, then for each n, $h_j|_{A_n}$ converges uniformly to $h|_{A_n}$.

REMARK 1. It should be stressed that the decomposition (A_n) is independent of the particular sequence (h_j) in H.

Let $H \subset \mathscr{L}$. We say that H satisfies the ⟨⟨*separation property*⟩⟩ if:

$$h_1 \in H,\ h_2 \in H,\ h_1 \neq h_2 \Rightarrow \tilde{h}_1 \neq \tilde{h}_2.$$

The relevant comment on the Generalized Egorov Theorem and the ⟨⟨*separation property*⟩⟩ is formulated in Theorem 2 below (see [**5**]; see also the notion of ⟨⟨partitionable function⟩⟩ introduced by M. Sion [**10**, p. 590]):

THEOREM 2. *Let $H \subset \mathscr{L}$ be compact metrizable for the topology of pointwise convergence on E. Then there is a set $E_0 \in \mathscr{E}$ carrying μ, with the following properties:*

(1) *For each $\varepsilon > 0$ there is a partition (A_n^ε) of E_0 with $A_n^\varepsilon \in \mathscr{E}$ and $\mu(A_n^\varepsilon) > 0$ for each n, such that:*

$$s \in A_n^\varepsilon,\ t \in A_n^\varepsilon,\ h \in H \Rightarrow |h(s) - h(t)| \leqq \varepsilon.$$

(2) *$H|_{E_0}$ satisfies the ⟨⟨separation property⟩⟩.*

We next make some remarks concerning the topology of pointwise convergence on a set of measurable functions:

REMARK 2. Let $H = \{h_1, h_2, \cdots, h_n, \cdots\}$ where each $h_n = 1_{B_n}$, with $B_n \in \mathscr{E}$. We may identify H with a subset of the compact space $\{0, 1\}^E$; the topology of pointwise convergence is then simply the product space topology. Various pathologies may occur:

(2.1) One may construct a sequence (h_n) such that *every* cluster value of this sequence is *non*-measurable (see for instance [**1**, Chapter IV (1952), p. 199, Exercise 4]).

(2.2) One may find $H = \{h_1, h_2, \cdots\}$ such that $\bar{H} = \{0, 1\}^E$.

Therefore the following questions are of interest:

Question 1. Let $H \subset \mathscr{L}$ be countable. Under what conditions is $\bar{H}$ compact metrizable (for the topology of pointwise convergence on E)?

Question 2. Let $H \subset \mathscr{L}$ be compact. Under what conditions is H metrizable (for the topology of pointwise convergence on E, of course)?

We shall begin with Question 2.

The following is a partial answer to Question 2, which however suffices for practical purposes (in Theorem 3 below we consider of course H endowed with the topology of pointwise convergence on E):

THEOREM 3 (METRIZATION CRITERION). *Let $H \subset \mathscr{L}$ be a set with the following properties*:

(i) *H is compact.*

(ii) *H is convex.*

(iii) *H satisfies the ⟨⟨separation property⟩⟩.*

Then H is metrizable.

To prove Theorem 3 one shows that, under our assumptions, the topology of pointwise convergence and the topology of convergence in probability coincide (see [**6**]). One may use in the proof the following remarkable theorem due to Komlós (see [7] or [**2**]):

THEOREM 4 (KOLMÓS). *Let (f_n) be a sequence of elements of $\mathscr{L}^1(E, \mathscr{E}, \mu)$ with $\sup_n \|f_n\|_1 < \infty$. Then one can find a subsequence (f_{n_k}) and an element $f \in \mathscr{L}^1(E, \mathscr{E}, \mu)$ such that (f_{n_k}), as well as any further subsequence extracted from (f_{n_k}), converges Cesàro to f, μ-almost surely.*

3. **An application to vector-valued mappings.** Let X be a *Banach space*, X' its *dual*. We denote the duality by $\langle x', x\rangle$, $x \in X$, $x' \in X'$. Let now $g: E \to X$. For $x' \in X'$ we denote by $\langle x', g\rangle$ the mapping $t \to \langle x', g(t)\rangle$ of E into R.

We recall that $g: E \to X$ is called *weakly measurable* if the real-valued mapping $\langle x', g\rangle$ is $\mathscr{E}$-measurable for each $x' \in X'$. We recall also that $g: E \to X$ is called *strongly* (Bochner) *measurable* if there is a sequence (s_n) of simple functions such that $\lim_n s_n(t) = g(t)$, μ-almost surely.

We may now state the following theorem (see [**6**]):

THEOREM 5 (WEAK VERSUS STRONG MEASURABILITY). *Let $g: E \to X$ be weakly measurable. We have*:

(1) *Suppose that the relations $x' \in X'$, $y' \in X'$ and $\langle x', g\rangle \neq \langle y', g\rangle$ imply $\langle x', g\rangle \not\equiv \langle y', g\rangle$. Then g is strongly measurable.*

(2) *Conversely, if $g: E \to X$ is strongly measurable, there is a set $E_0 \in \mathscr{E}$ carrying μ such that the relations $x' \in X'$, $y' \in X'$ and $\langle x', g\rangle|_{E_0} \neq \langle y', g\rangle|_{E_0}$ imply $\langle x', g\rangle \not\equiv \langle y', g\rangle$.*

It appears, therefore, that the ⟨⟨separation property⟩⟩ really makes the difference between weak measurability and strong measurability.

4. **The ⟨⟨separation property⟩⟩ and the notion of lifting.** The most *convenient* way to obtain the ⟨⟨separation property⟩⟩, at least for sets of bounded measurable functions, is by applying the *notion of lifting*:

We recall that a mapping $\rho: \mathscr{L}^\infty \to \mathscr{L}^\infty$ is called a *lifting of $\mathscr{L}^\infty$* if it satisfies the following conditions:

(I) $\rho(f)\equiv f$;

(II) $f\equiv g$ implies $\rho(f)=\rho(g)$;

(III) $\rho(1)=1$;

(IV) $\rho(af+bg)=a\rho(f)+b\rho(g)$;

(V) $\rho(fg)=\rho(f)\rho(g)$.

Without going into the history of the subject, it suffices to recall that if $(E, \mathscr{E}, \mu)$ is a *complete probability space*, then a lifting of $\mathscr{L}^\infty$ always exists (see for instance [**4**]).

Henceforth we assume that $(E, \mathscr{E}, \mu)$ is a complete probability space.

There is an equivalent way of defining the notion of lifting if one prefers to work with sets rather than functions. For each $A\in\mathscr{E}$, $\rho(1_A)$ is again an indicator function (by axiom (V)); we write

$$(*)\qquad \rho(1_A) = 1_{\rho(A)}.$$

The mapping $\rho:\mathscr{E}\to\mathscr{E}$ obtained in this manner satisfies the conditions:

(I′) $\rho(A)\equiv A$;

(II′) $A\equiv B$ implies $\rho(A)=\rho(B)$;

(III′) $\rho(E)=E$, $\rho(\varnothing)=\varnothing$;

(IV′) $\rho(A\cup B)=\rho(A)\cup\rho(B)$;

(V′) $\rho(A\cap B)=\rho(A)\cap\rho(B)$.

The mapping $\rho:\mathscr{E}\to\mathscr{E}$ satisfying axioms (I′)–(V′) is called a *lifting* of $\mathscr{E}$. Since this cannot lead to confusion, we shall use the same notation for the lifting of $\mathscr{L}^\infty$ and the corresponding lifting of $\mathscr{E}$.

Lifting topology. Let now ρ be a fixed lifting of $\mathscr{L}^\infty=\mathscr{L}^\infty(E, \mathscr{E}, \mu)$. Corresponding to the lifting ρ we may introduce a topology $\mathscr{T}_\rho$ on the space E as follows:

$$\mathscr{T}_\rho = \{\rho(A) - N \mid A\in\mathscr{E}, N\in\mathscr{E}, \mu(N)=0\}.$$

The topology $\mathscr{T}_\rho$ turns out to have the following properties (see [**4**, p. 59]):

(1) $\mathscr{T}_\rho$ is extremally disconnected.

(2) $C_R^b(E, \mathscr{T}_\rho)=\{\rho(g)\mid g\in\mathscr{L}^\infty\}$.

We may now give an answer to Question 1 raised in §2.

We shall only consider the case of a bounded set $H\subset\mathscr{L}^\infty$. We have the following analogue of Arzela-Ascoli's theorem (see [**5**], [**6**]):

THEOREM 6. *Let $H\subset\mathscr{L}^\infty$ be a bounded set. We have:*

(1) *Suppose that H is compact metrizable for the topology of pointwise convergence on E. There is then a set $E_0\in\mathscr{E}$ carrying μ, such that $H|_{E_0}\subset C_R^b(E, \mathscr{T}_\rho)|_{E_0}$ and $H|_{E_0}$ is equicontinuous on (the $\mathscr{T}_\rho$-open set) E_0 with respect to $\mathscr{T}_\rho$.*

(2) *Conversely, suppose that $H\subset C_R^b(E, \mathscr{T}_\rho)$ and that H is equicontinuous*

with respect to $\mathscr{T}_\rho$. *Then* $\bar{H}$ *(closure of H for the topology of pointwise convergence* on *E) is compact metrizable.*

Another application. Let $(E, \mathscr{E}, \mu)$ be a complete probability space and Z a completely regular topological space.

We recall the definition of the *abstract space* $\mathscr{L}_Z^\infty = \mathscr{L}_Z^\infty(E, \mathscr{E}, \mu)$. A mapping $f: E \to Z$ belongs to $\mathscr{L}_Z^\infty$ if:

(i) $f(E) \subset Z$ is relatively compact;

(ii) $f: E \to Z$ is weakly measurable, that is, $h \circ f$ is $\mathscr{E}$-measurable, for each $h \in C_R(Z)$.

It is clear that if $Z = R$, then $\mathscr{L}_R^\infty = \mathscr{L}^\infty$.

Let now ρ be a *lifting of* $\mathscr{L}^\infty$. Starting with ρ one may define an ⟨⟨abstract lifting⟩⟩ of the abstract space $\mathscr{L}_Z^\infty$ as follows: For $f \in \mathscr{L}_Z^\infty$ we set

$$h \circ \rho_Z(f) = \rho(h \circ f), \quad \text{for all } h \in C_R(Z).$$

The above "weak invariance formula" uniquely determines the abstract lifting ρ_Z associated with ρ (see [**4**, pp. 52–53]). Since there can be no confusion, we shall denote this abstract lifting by ρ again.

This notion of abstract lifting has many advantages: let us mention in passing that it permits to give a very simple and rapid proof of Doob's classical theorem concerning the "existence of a separable modification" of a stochastic process (see [**3**] or [**4**]).

Let us now consider again a Banach space X and let us return to weakly measurable versus strongly measurable mappings.[2]

Consider $(X, \sigma(X, X'))$ and correspondingly the abstract space $\mathscr{L}^\infty_{(X,\sigma(X,X'))}$.

We note that $f \in \mathscr{L}^\infty_{(X,\sigma(X,X'))}$ if

(i) $f(E) \subset X$ is $\sigma(X, X')$-relatively compact.

(ii) $\langle x', f\rangle$ is $\mathscr{E}$-measurable, for each $x' \in X'$.

We then have (see [**5**]):

THEOREM 7. *Let* $f \in \mathscr{L}^\infty_{(X,\sigma(X,X'))}$ *and let* $g = \rho(f)$. *Then*

$$\rho(\langle x', g\rangle) = \langle x', g\rangle, \quad \textit{for each } x' \in X'$$

and hence g is strongly measurable.

It is enough to note that since g satisfies the above invariance property, g has the separation property required in Theorem 5, and therefore g is strongly measurable.

[2] Below we denote by $\sigma(X, X')$ the X'-topology on X, that is the weakest topology on X making every linear functional $x' \in X'$ continuous.

REMARK 3. The fact that if $f:E \to X$ is a weakly measurable mapping taking values in a $\sigma(X, X')$-compact set, then its ⟨⟨weak equivalence class⟩⟩ contains a strongly measurable mapping is the object of a classical theorem due to R. S. Phillips [**9**]. The known proofs introduce the associated weakly compact operator of L^1_R into X (see [**1**, Chapter 6 (1959), p. 95, Exercise 25], or [**4**, pp. 91–92]), or the associated vector-valued measure (see [**11**, pp. 115–118]) and are quite laborious. The separation property, as exhibited in Theorem 5, makes the notion of lifting appear as a natural tool in this type of problem: We obtain a one-line proof of Phillips' theorem. Furthermore the notion of lifting yields a canonical way of constructing a strongly measurable function g in the weak equivalence class of f.

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Current address: Department of Mathematics, Northwestern University, Evanston, Illinois 60201