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SOBOLEV INEQUALITIES FOR RIEMANNIAN BUNDLES

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1. **Introduction.** Sobolev inequalities play a major role in the study of differential operators and nonlinear functional analysis. The inequalities are the primary tools in the study of the properties of spaces of functions with Sobolev topologies; for example, the Schauder ring theorem. There are theorems involving continuity and closure of composition in such spaces [**3**, Chapter 2]. The latter theorems involve application of the inequalities to vector fields. It is this case and its generalization which this paper studies, where $\boldsymbol{R}^n$ is replaced by an arbitrary Riemannian manifold satisfying certain geometric conditions.

While the Sobolev inequalities over $\boldsymbol{R}^n$ have been known for some time, the usual proofs use transform methods and are therefore hard to generalize. In 1959 Nirenberg [**4**] presented particularly elegant proofs, due to himself and other authors, which could be generalized. These proofs are the basis for the results of this paper.

Throughout, M denotes a complete Riemannian n-dimensional manifold without boundary. The canonical volume form on M is denoted dV. Let $\pi: E \to M$ be a vector bundle with a specified smooth metric $(\ ,\ )$. ∇ is a connection on E satisfying $d(V, W)x_m = (\nabla_{x_m} V, W) + (V, \nabla_{x_m} W)$, where V and W are sections of E and $x_m \in T_m M$. ∇^n is the iterated covariant derivative. In most applications E is a tensor bundle over M.

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$C_0^\infty(E)$ is the space of C^∞ sections of E with compact support. For $x \in M$, $V \in C_0^\infty(E)$, the quantity $|\nabla^i V(x)|$ is the norm of $\nabla^i V(X)$ in the canonical norm of $L^i(TM:E)$.

A more detailed account of the proofs can be found in [3] and shall be published elsewhere. The author wishes to thank Jerrold Marsden for his help and encouragement and E. Calabi for his assistance.

2. **Statement of results.**

DEFINITION 1. Let $V \in C_0^\infty(E)$. Then define

(i) (C^k norm) $\|V\|_k = \sum_0^k \sup|\nabla^i V(x)|$;

(ii) (L_k^p, Sobolev norm) $|V|_{p,k} = \sum_0^k (\int_M |\nabla^i V(x)|^p \, dV)^{1/p}$;

(iii) (Holder norm) for $0<\theta<1$,

$$[V]_{\theta,k} = \sum_0^k \sup_{x,y\in M} \sup_{C\in G(x,y)} \frac{|\tau(C)\nabla^i V(x) - \nabla^i V(y)|}{d(x,y)^\theta},$$

where $G(x,y) = \{$length-minimizing geodesics joining x and $y\}$, $\tau(C)$ is parallel translation along C from $\pi^{-1}(y)$ to $\pi^{-1}(x)$, and $d(x,y)$ is the distance from x to y.

Each of the functions defined in Definition 1 is a norm on the vector space $C_0(E)$.

Also, denote $\| \;\|_0 = \| \;\|$, $| \;|_{p,0} = | \;|_p$, and $[\;]_{\theta,0} = [\;]_\theta$.

DEFINITION 2. (i) $C^k(E)$ (resp. $L_k^p(E)$, $C^{k+\theta}(E)$) is the completion of $C_0^\infty(E)$ with respect to $\| \;\|_k$ (resp. $| \;|_{p,k}$, $[\;]_{\theta,k}$).

We note that if $0<\theta<1$, then $C^{k+1}(E) \subset C^{k+\theta}(E) \subset C^k(E)$ and the inclusion is continuous.

We state the following hypotheses:

(C1) The injective radius of M is bounded away from zero.

(C2) There is a δ such that for each $x \in M$ and $V, W \in T_xM$, the sectional curvature $|K_x(V, W)| < \delta$.

THEOREM 1. *Let M satisfy* C1 *and* C2. *Then if $p>1$ and $s>n/p+k$, there is a constant C such that, for all $f \in L_s^p(E)$, we have $\|f\|_k \leqq C|f|_{p,s}$, where C is independent of f.*

THEOREM 2. *Let M satisfy* C1 *and* C2 *and assume $r>n$. Then there is a constant C such that, for each $f \in L_s^r(E)$, we have $[f]_{1-n/r,s} \leqq C|f|_{r,s+1}$, where C is independent of f.*

THEOREM 3. *Let M satisfy* C1 *and* C2, $0 \leqq j \leqq m$, *and $q, r \geqq 1$. Then if $m/p = j/r + (m-j)/q$, there is a constant C such that, for each $f \in L_m^r(E) \cap L^q(E)$, we have $|\nabla^j f|_p \leqq C|f|_{r,m}^{j/m} |f|_q^{1-j/m}$, where C is independent of f.*

In the next theorem, we adopt the following notation, due to Nirenberg [4].

Notation. For $p>0$, the definition of $|\ |_p$ remains as in Definition 1. For $p<0$ and $f\in C_0^\infty(E)$, let $h=[-n/p]$, $-\alpha=h+n/p$ and

$$|f|_p = \|\nabla^h f\|, \qquad \alpha = 0,$$
$$|f|_p = [\nabla^h f]_\alpha, \qquad \alpha > 0.$$

THEOREM 4. *Let M satisfy* C1 *and* C2, *and let $0\leqq j\leqq m$. Then if $m-j-n/r$ is not a nonnegative integer and $1/p=1/r+(j-m)/n$, there is a constant C such that, for $f\in L_m^r(E)$, it follows that $|\nabla^j f|_p\leqq C|\nabla^j f|_{r,m-j}$ where C is independent of f.*

Under some circumstances, one can interpolate between these inequalities. We do have the standard interpolation lemma:

LEMMA. *If $-\infty<\lambda\leqq\mu\leqq\nu<\infty$ and $\lambda\geqq 0$ or $\nu\leqq 0$, then, for all $u\in L^{1/\lambda}(E)\cap L^{1/\nu}(E)$, $u\in L^{1/\mu}(E)$,*

$$|u|_{1/\mu} \leqq C\,|u|_{1/\lambda}^{(\nu-\mu)/(\nu-\lambda)}\,|u|_{1/\gamma}^{(\mu-\lambda)/(\nu-\lambda)}.$$

In the classical case the assumption that λ, μ, and ν all have the same sign is unnecessary [**4**, p. 126]. Under what conditions it is necessary is not known to the author.

There are several applications of this lemma. For example,

THEOREM 5. *Let M satisfy* C1 *and* C2 *and let $1\leqq q, r\leqq\infty$ and $0\leqq j\leqq m$. If $j/n+1/r-m/n\geqq 0$ and, for $j/m\leqq a\leqq 1$,*

$$\frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + (1-a)\frac{1}{q},$$

then, for $u\in L_m^r(E)\cap L^q(E)$, it follows that

$$|\nabla^j u|_p \leqq C\,|u|_{r,m}^a\,|u|_q^{1-a},$$

where C is independent of u.

3. **Discussion of the proofs.** The fundamental idea behind all of the theorems is the reduction of the argument to a local one. Using condition C2 and standard comparison results [**1**, pp. 250–257], one gets the necessary uniform bounds on the exp maps. Note that C2 implies an upper bound in the Ricci curvatures. Thus if $2R$ is the injective radius of M, for $V\in T_xM$, $|V|<R$, it follows that the Jacobian $\mathrm{Jexp}_x V>A$, where A is independent of x and V.

PROOF OF THEOREM 1. All constants are denoted C throughout. We assume $k=0$; the general case follows by induction. Let $x\in M$. With $2R$ the injective radius of M, let $B_x(R)$ be the ball of radius R in T_xM. Let (r,θ) be spherical coordinates on T_xM.

Let $g:\boldsymbol{R}\to\boldsymbol{R}$ be a C^∞ function such that $g(t)1$ for $t<R/2$ and $g(t)=0$, $t\geqq 3R/4$, and $|g^{(i)}(t)|<A$ for $i\leqq s$. Fix θ and let $\tau(\theta)$ be parallel translation along the geodesic $r\to\exp_x(r,\theta)$. From

$$u(x) = -\int_0^R \frac{\partial}{\partial r}(\tau(\theta)g(r))u(\exp_x(r,\theta))\,dr,$$

using integration by parts, we get

$$|u(x)| = \frac{(-1)^{s-1}}{(s-1)!}\int_0^R r^{s-1}\left|\frac{\partial^s}{\partial r^s}(\tau(\theta)g(r))u(\exp_x(r,\theta))\right|dr.$$

From the standard formula relating τ to the covariant derivative,

$$|u(x)| \leqq C\int_0^R r^{s-1}|\nabla^s g(r)u(\exp(r,\theta))|\,dr.$$

Integrate with respect to $r^{n-1}\,dS$ on the unit sphere in T_xM. Thus

$$|u(x)| \leqq C\int_{s_\theta}\int_0^R r^{s-1}|\nabla^s g(r)u(\exp_x(r,\theta))|\,r^{n-1}r\,dS.$$

Now $dV'=r^{n-1}\,dr\,dS$ is the volume in T_xM. Apply Holder's inequality and conclude

$$|u(x)| \leqq C\left(\int_{s_\theta}\int_0^R r^{(s-n)p/(p-1)}r^{(n-1)}\,dr\,dS\right)^{p-1/p}$$
$$\times\left(\int_{B_m(R)}|\nabla^s g(r)u(\exp(r,\theta))|^p\,dV'\right)^{1/p}.$$

Since $s>n/p$, the first integral is finite and its value is independent of x. Now apply the product rule for covariant derivatives and Minkowski's inequality to the second integral to conclude

$$|u(x)|^p \leqq C\int_{B_m(R)}|\nabla^s u|^p\,dV'.$$

Now if dV is the volume element on M then on $\exp_x(B_m(R))$, we have $dV(\exp(r,\theta))=\mathrm{Jexp}(r,\theta)\,dV'$. Thus, by the remark preceding the proof

$$|u(x)|^p \leqq C\int_{\exp(B_m(R))}|\nabla^s u|^p\,dV \leqq \int_M |\nabla^s u|^p\,dV.$$

Since this holds for each $x\in M$, the theorem follows. Q.E.D.

Theorem 2 is proven similarly to Theorem 1.

Theorems 3 and 4 depend on the existence of a smooth triangulation with known limits on the mesh subordinate to a cover by normal neighborhood. Conditions C1 and C2 guarantee such a triangulation [**2**]. This is used to construct a C^k uniform collection of partition functions. Using these one can reduce the theorems to the case where the sections have compact support in some normal neighborhood in M. In each theorem this case is handled by a suitable generalization of the arguments in [**4**]. However, since the derivatives of the partition functions cannot be ignored, the right side of the inequality contains entire L^p_k norms rather than the L^p norms of the kth derivative, as in the classical case.

The induction argument used in each of the theorems goes through almost exactly as in the classical case, simply noting that if f is a smooth section of E, then ∇f is a smooth section of the bundle $L(TM, E)$ which has a canonical metric and connection.

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