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STABLE HOMOTOPY THEORY OVER A FIXED BASE SPACE

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A spectral sequence which may be regarded as an 'Adams spectral sequence over a fixed space B' has been constructed by J. F. McClendon **[6]** and J.-P. Meyer **[8]**. This note describes a generalization in which there is no need for any orientability assumptions. The construction is carried out in a suitable stable category, which may be of independent interest. An application to the enumeration of immersions is given. Details will appear elsewhere.

1. **A stable category.** Let $Ex\text{-}B$ denote the category of ex-spaces (in the terminology of **[4]**) of the path-connected complex B. It is known that the corresponding homotopy category $(Ex\text{-}B)_h$ exhibits certain stability properties **[4**, Theorem 6.4]. One can construct a category $\mathscr{S}/B$, in which the corresponding stable homotopy theory can be investigated, by formalizing the notion of a 'bundle' over B with fibre a CW spectrum (in the sense of **[3]**, **[11]**). The details are as follows. If F_1, F_2, F are objects of the category $\mathscr{S}$ of CW spectra, there is a simplicial set of morphisms $\mathrm{Mor}_{\mathscr{S}}(F_1, F_2)$; and $\mathrm{Mor}_{\mathscr{S}}(F, F)$ is a simplicial monoid whose invertible elements form a simplicial group $\mathrm{Aut}_{\mathscr{S}}F$. We take B to be a simplicial set rather than a space.

DEFINITION. An *object* of $\mathscr{S}/B$ is a pair (F, ξ) where $F \in \mathrm{ob}\,\mathscr{S}$ and ξ is a principal simplicial $\mathrm{Aut}_{\mathscr{S}}\, F$-bundle over B.

A *morphism* from (F_1, ξ_1) to (F_2, ξ_2) is a section of the simplicial bundle with fibre $\mathrm{Mor}_{\mathscr{S}}(F_1, F_2)$ associated to the principal $(\mathrm{Aut}_{\mathscr{S}}F_1 \times \mathrm{Aut}_{\mathscr{S}}F_2)$-bundle $\xi_1 \times \xi_2$ over B.

This category inherits much of the usual machinery of stable homotopy theory from Boardman's category $\mathscr{S}$; for example, it has an invertible translation-suspension functor S_B, arbitrary wedges, and smash-product functors. The corresponding homotopy category $(\mathscr{S}/B)_h$ is additive, and triangulated with respect to S_B, and the axioms of Puppe **[9]** hold. There is a *stabilization functor* $(Ex\text{-}B)_h \to (\mathscr{S}/B)_h$ which is bijective on morphism-sets $[X, Y]_B$ of $(Ex\text{-}B)_h$ whenever X is a relative CW ex-space of B, Y is an ex-space fibred over B (with fibre F, say) and $\dim(X-B) \leqq 2\,\mathrm{conn}\, F$.

2. **A cohomology theory on $\mathscr{S}/B$.** Let p be a prime, and let $\rho: \pi_1 B \to GL(V)$ be a semisimple representation of $\pi_1 B$ on a finite-dimensional vector

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space V over Z/pZ. Then ρ gives rise to a system of twisted coefficients on B, and hence to a cohomology theory on *Ex-B*. We define a corresponding cohomology functor on $\mathscr{S}/B$ by constructing a representing object $K(\rho) \in \mathrm{ob}(\mathscr{S}/B)$. This object is a 'bundle' with fibre the Eilenberg-Mac Lane spectrum $K(V) \in \mathrm{ob}\, \mathscr{S}$, and is a stable version of the Eilenberg-Mac Lane bundles in **[10]**.

As the analogue 'over B' of the Steenrod algebra A_p^*, we obtain a graded abelian category $A_{B,p}^*$ as follows. The objects are the finite-dimensional, semisimple representations of $\pi_1 B$ over the field Z/pZ: the morphisms from a representation ρ to a representation σ form the graded abelian group $\{K(\rho), K(\sigma)\}_B^*$, where $\{\ ,\ \}_B^*$ denotes graded homotopy classes in $\mathscr{S}/B$. Composition is evident. We regard $A_{B,p}^*$ as a 'ring with several objects', and introduce the corresponding abelian category of graded left modules $A_{B,p}^*$*-mod*: this is just the category of additive functors from $A_{B,p}^*$ to the category of graded abelian groups.

The functors on $\mathscr{S}/B$ represented by the various $K(\rho)$ now unite to give a cohomology functor

$$\boldsymbol{H}^*(\ \ ; p) : (\mathscr{S}/B)_h \to A_{B,p}^*\text{-}mod.$$

When B is simply-connected, $A_{B,p}^*$*-mod* is equivalent to the category of graded left modules over the Massey-Peterson semitensor algebra $H^*(B; Z/pZ) \bigcirc A_p^*$ [7].

3. **A spectral sequence.** We wish to use the foregoing cohomology theory to construct an Adams spectral sequence for $\{X, Y\}_B^*$, where X and Y are bundles of spectra over B. As in the classical Adams spectral sequence, it is necessary to impose some finiteness assumption on X, and finite type assumption on Y, to ensure existence and convergence of the spectral sequence. We shall assume the following:

(i) $\pi_1 B$ has only finitely many distinct irreducible finite-dimensional representations over Z/pZ.

(ii) For each such irreducible ρ, the groups $H^*(B; \rho)$ and $H^*(Y; \rho)$ are finitely-generated in each dimension.

(iii) The fibre-spectrum F of Y is highly-connected, and $\pi_* F$ is finitely-generated in each dimension.

(iv) X is the image in $\mathscr{S}/B$ of a finite (relative) CW ex-space of B.

Under these conditions, there is a diagram in $\mathscr{S}/B$ with exact triangles (dotted morphisms have degree -1)

$$\begin{tikzcd}
Y = Z_0 \arrow[dr] & & Z_1 \arrow[ll] \arrow[dr] & & Z_2 \arrow[ll] \arrow[dr] & & Z_3 \arrow[ll] \arrow[dr] & & \arrow[ll, dashed] \\
& K_0 \arrow[ur, dashed] & & K_1 \arrow[ur, dashed] & & K_2 \arrow[ur, dashed] & & {}
\end{tikzcd}$$

such that the induced cohomology diagram

$$H^*(Y;p) \leftarrow H^*(K_0;p) \leftarrow H^*(K_1;p) \leftarrow \cdots$$

is a minimal free resolution of $H^*(Y;p)$ in $A^*_{B,p}$-*mod.* We now obtain a Cartan-Eilenberg system by applying the functor $\{X,\ \}^*_B$, and hence a spectral sequence.

THEOREM. *Under the conditions* (i)–(iv) *above, this spectral sequence has*

$$E_2^{s,t} \approx \operatorname{Ext}^{s,t}_{A^*_{B,p}\text{-}mod}(H^*(Y;p), H^*(X;p))$$

and converges to the quotient of $\{X, Y\}^*_B$ *by the subgroup of torsion elements of order prime to* p.

The proof uses the existence of nonsimple modified Postnikov towers **[10]**, and otherwise follows the convergence proof of **[1]**.

Smash- and composition-product pairings can be introduced into the spectral sequence. The former can be used to give it the structure of a module over the usual Adams spectral sequence for the stable stems.

4. **The enumeration of immersions.** By the immersion theory of M. Hirsch, the regular homotopy classes of smooth immersions of a smooth manifold M^m in R^{m+n} are enumerated for $n>1$ by liftings in the diagram

$$\begin{array}{ccc} & & BO_n \\ & & \downarrow \\ M^m & \xrightarrow{\nu} & BO \end{array}$$

where ν represents the stable normal bundle. According to a theorem of J. C. Becker [2], in the case $m \leqq 2n-2$ these liftings are enumerated by difference classes in a certain stable track group in *Ex-BO*, which becomes a morphism group in $(\mathscr{S}/BO)_h$ and hence can be calculated from the spectral sequence. Calculation with specific resolutions for the spectrum bundle obtained from $BO_n \to BO$ yields the following results for real projective spaces.

THEOREM. *Suppose* P^m *immerses in* R^{2m-k}, *where* $0 \leqq k \leqq 5$ *and* $(m-k) \geqq 7$. *Then the difference group for such immersions is as given in Table* 1. *In particular, the number of regular homotopy classes is given by the order of the appropriate group.*

This theorem extends special cases of results of I. M. James and P. E. Thomas [5], and appears to agree with recent calculations of H. A. Salomonsen, who uses a more geometric approach. The ambiguities in the table are due to undetermined higher differentials.

m \ k	0	1	2	3	4	5
$8q$	Z	Z_2	0	0	0	Z_8 or Z_4
$8q+1$	Z_2	$Z_2+Z_2+Z_2$	Z_2+Z_2	$Z_8+Z_2+Z_3$	0	0
$8q+2$	Z	Z_4	Z_2	Z_2	0	Z_8 or Z_4
$8q+3$	Z_2	Z_4+Z_2	Z_8+Z_4	$Z_8+Z_8+Z_4+Z_4+Z_2+Z_3$	Z_8+Z_8	Z_8+Z_4
$8q+4$	Z	Z_2	0	0	0	Z_8 or Z_4
$8q+5$	Z_2	$Z_2+Z_2+Z_2$	Z_2+Z_2	$Z_8+Z_2+Z_3$	0	0
$8q+6$	Z	Z_4	Z_2	Z_2+Z_2	Z_2+Z_2	$Z_8+Z_2+Z_2$ or $Z_4+Z_2+Z_2$
$8q+7$	Z_2	Z_4+Z_2	Z_8+Z_4	$Z_{16}+Z_4+Z_4+Z_4+Z_2+Z_3$	$Z_{16}+Z_8+Z_2$	$Z_{16}+Z_8+Z_2$

Table 1. Difference groups of immersions of RP^m in R^{2m-k} $(m-k\geq 7)$.

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