PROPERTIES OF THREE ALGEBRAS RELATED TO $L^p$-MULTIPLIERS

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1. Introduction. In this paper we shall announce several properties of certain algebras which arise in the study of $L^p$-multipliers; detailed proofs will be given elsewhere. Let $G$ be a locally compact abelian group and let $\Gamma$ denote its dual group. Let $L_p(\Gamma)$ denote the space of $p$-integrable functions on $\Gamma$ with respect to Haar measure, and let $q$ denote the index which is conjugate to $p$. Let

$$A_p(\Gamma) = [L_p(\Gamma) \otimes L_q(\Gamma)]/K$$

where $K$ is the kernel of the convolution operator $c: L_p \otimes L_q(\Gamma) \rightarrow C_0(\Gamma)$ by $c(f \otimes g)(\gamma) = (f \ast g)(\gamma)$. $A_p(\Gamma)$ is the $p$-Fourier algebra which was introduced by Figa-Talamanca in [6] where it was shown that $A_p(\Gamma)^*$ is isometrically isomorphic to $M_p(\Gamma)$, the bounded, translation invariant, linear operators on $L_p(\Gamma)$. Herz [11] showed that $A_p(\Gamma)$ is a Banach algebra under pointwise multiplication; it is known that $A_2(\Gamma) = A(\Gamma) = L_1(G)^\wedge$ and that the inclusions $A_2(\Gamma) \subset A_p(\Gamma) \subset A_1(\Gamma) = C_0(\Gamma)$ are norm decreasing if $1 < p < 2$; see [5], [6], [11] for the basic properties of $A_p(\Gamma)$.

Let $B_p(\Gamma)$ denote the algebra of continuous functions $f$ on $\Gamma$ such that $f(\gamma)h(\gamma) \in A_p(\Gamma)$ whenever $h \in A_p(\Gamma)$. The multiplier algebra $B_p(\Gamma)$ is a Banach algebra in the operator norm. We have studied $B_p(\Gamma)$ in [8], [9].

Fix $p$ in $1 < p < 2$.

Regard $L_p(\Gamma)$ as an algebra of convolution operators on $L_p(\Gamma)$ and let $m_p(\Gamma)$ denote the closure of $L_1(\Gamma)$ in $M_p(\Gamma)$. The first result of this paper says that $B_p(\Gamma)$ is isometrically isomorphic to the dual space $m_p(\Gamma)^*$. In the second result, we use certain properties of $B_p(\Gamma)$ to give a theorem of Eberlein type for $m_p(\Gamma)$. In the final section of the paper, we use

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m_p(\Gamma) to represent M_p(\Gamma) as the multiplier algebra of a certain subalgebra of M_p(\Gamma). For the case when \Gamma is a compact group, Theorem 1 was proved in [1].

Our work has been influenced by Theorems 1.1 and 1.2 of [12]. There McKilligan studied the multiplier algebra B of a commutative, semisimple, Banach algebra A which contains an approximate identity of norm one. A** is equipped with the Arens product (\cdot) and B is isometrically embedded in (A**, \cdot) by the mapping T \mapsto T**(j) where j is the right identity in A**; see [2] for the basic properties of the Arens product. Thus if \( T \in B \) and if \( \{e_\alpha\} \) is the approximate identity in A, then

\[ T**(j)(F) = \lim_{\alpha} F(T(e_\alpha)) \]

for every functional \( F \in A^* \).

We shall not distinguish between \( M_p(\Gamma) \) and \( A_p(\Gamma)^* \). If \( H \in L_1(\Gamma) \), let \( *H \) denote the corresponding convolution operator on \( L_p(\Gamma) \). If \( \psi \in m_p(\Gamma)^* \), let \( \|\psi\|_* \) denote the norm of \( \psi \). If \( h \in A_p(\Gamma) \), \( |h|_p \) denotes its norm; if \( f \in B_p(\Gamma) \), \( \|f\|_p \) is the operator norm; and if \( T \in M_p(\Gamma) \), \( \|T\|_p \) is the operator (or functional) norm of \( T \). An approximate identity \( \{E_\alpha\} \) in \( L_1(G) \) which consists of normalized characteristic functions of compact, symmetric neighborhoods of the identity is referred to as a standard approximate identity. The corresponding net \( \{E_\alpha\} \) in \( A_2(\Gamma) = A(\Gamma) \) is also referred to as a standard approximate identity.

After we had submitted this paper for publication, Professor Carl Herz told us that he had given a different proof of Theorem 1 in 1970 for amenable groups. N. Lohoué, C.R. Acad. Sci. Paris 272 (1971) and 273 (1971), refers to Herz’s result as presented at Orsay in June, 1970.

2. Dual space representation.

**Theorem 1.** \( B_p(\Gamma) \) is isometrically isomorphic to \( m_p(\Gamma)^* \) by the map \( \varphi \mapsto \tilde{\varphi} \) when \( \tilde{\varphi}(*H) = \int_{\Gamma} \varphi(\gamma)H(\gamma) \, d\gamma \) for all \( H \in L_1(\Gamma) \).

Use Theorems 1 of [6] and [7] to show that \( h \mapsto \tilde{h} \) gives an isometric embedding of \( A_p(\Gamma) \) into \( m_p(\Gamma)^* \). Use a standard approximate identity to extend this embedding to \( B_p(\Gamma) \). Conversely, let \( \tilde{\varphi} \in m_p(\Gamma)^* \); then there is a bounded measurable function \( \psi_0(\gamma) \) such that

\[ \tilde{\varphi}(*H) = \int_{\Gamma} \psi_0(\gamma)H(\gamma) \, d\gamma. \]

Define

\[ \psi_{ab}(\gamma) = (\psi_0E_\alpha) * f_\beta(\gamma) \]

when \( \{E_\alpha\} \) and \( \{f_\beta\} \) are standard approximate identities in \( L_1(G) \) and \( L_1(\Gamma) \) respectively. Then \( \|\psi_{ab}\|_p \leq \|\tilde{\varphi}\|_* \) and \( \{\psi_{ab}\} \) converges to \( \psi_0 \) in the weak*
topology of $L^\infty(\Gamma)$. Let $\mathfrak{B}_p$ denote the algebra of bounded measurable functions $\psi$ on $\Gamma$ for which $M(\psi)(x, y) := \psi(xy^{-1})$ is a multiplier on $L_p \otimes L_q(\Gamma)$. By following Eymard [5], one shows that $\mathfrak{B}_p(\Gamma) = B_p(\Gamma)$. Let $E_\theta = L_p \otimes_\lambda L_q(\Gamma)$, the completion of $L_p \otimes L_q(\Gamma)$ with respect to the least cross norm. By a theorem of Grothendieck [10, p. 122], $E_\theta^* = L_p \otimes L_q(\Gamma)$. Using this fact one shows that $M(\psi_\delta)$ converges to $M(\psi_0)$ in the weak* topology of $L_p \otimes L_q(\Gamma)$ and that $\psi_0 \in \mathfrak{B}_p(\Gamma)$.

By letting $m_p(\Gamma)^* = M(\delta)^*/m(\delta)$ have the quotient Arens product, one sees that $\varphi \mapsto \tilde{\varphi}$ is an algebra isomorphism as well.

3. Eberlein’s theorem. Use McKilligan’s representation for multipliers to regard a function $f \in B_p(\Gamma)$ as a functional $\tilde{f} \in M_p(\Gamma)^*$.

**Theorem 2.** Let $M_p(\Gamma)_c$ denote the $L^p$-multipliers with continuous Fourier transforms. An operator $T \in M_2(\Gamma)_c$ is in $M_p(\Gamma)_c$ if and only if there is a constant $M \geq 0$ such that for every finite set $\{a_k\}$ of complex numbers and every equinumerous subset $\{g_k\} \subset G$, the Fourier transform $\tilde{T}$ of $T$ satisfies

$$\left| \sum_{k=1}^n a_k \tilde{T}(g_k) \right| \leq \frac{M}{\sum_{k=1}^n a_k \tilde{g}_k}_p.$$

When $T \in M_p(\Gamma)_c$, $\|T\|_p$ is the least constant $M$ for which the inequality holds.

If $T \in M_p(\Gamma)_c$, it follows from McKilligan’s representation that $\tilde{g}(T) = \tilde{T}(g)$ for $g \in G$, so that the inequality holds for some $M \leq \|T\|_p$. By Saeki’s Theorem 4.3 of [14], $\|T\|_p$ is the least constant $M$ for which the inequality holds. If $T \in M_2(\Gamma)_c$ satisfies the inequality, so does $T_{\delta\theta} = * (f_\delta T(E_\theta))$ when $\{f_\delta\} \subset L_1(G)$ and $\{E_\theta\} \subset L_1(\Gamma)$ are standard approximate identities. Since $\|T_{\delta\theta}\|_p \leq M$, the net $\{T_{\delta\theta}\}$ has a weak* convergent subnet $\{T_\delta\}$ in $M_p(\Gamma)$. Since $A_2(\Gamma) = A(\Gamma)$ is dense in $A_p(\Gamma)$, it follows that $T = \lim T_\delta$ is in $M_p(\Gamma)$.

From [14], a function $F \in L^\infty(G)$ is said to be regulated if there is an approximate identity $\{E_\delta\}$ of norm one in $L_1(G)$ such that $\{F \ast E_\delta\}$ converges pointwise and in the weak* topology of $L^\infty(G)$ to $F$. Theorem 2 can be extended so as to apply to operators with regulated Fourier transforms.

The separation theorem [3, p. 417] for compact convex sets and Theorem 2 now imply

**Theorem 3.** If $f \in B_p(\Gamma)$, there is a net $\{f_\delta\}$ in the span of $G$ in $B_p(\Gamma)$ such that $\|f_\delta\|_p \leq \|f\|_p$ and such that $\{f_\delta\}$ converges to $f$ in the weak* topology of $B_p(\Gamma)$. 
4. \( M_p \) as a multiplier algebra. Use multiplication of operators to regard \( M_p(\Gamma) \) as an algebra over the ring \( m_p(\Gamma) \). In particular, \( M_p(\Gamma) \) is an \( m_p(\Gamma) \)-module. It follows from the general form of Cohen's factorization theorem \[13, \text{p. 453}\] that the \( m_p \)-essential submodule of \( M_p(\Gamma) \) is

\[
M_p m_p(\Gamma) = \{ K \in M_p(\Gamma) \mid K = UT, U \in M_p(\Gamma), T \in m_p(\Gamma) \}.
\]

\( M_p m_p(\Gamma) \) is a Banach algebra in the operator norm and a standard approximate identity in \( L_1(\Gamma) \) is an approximate identity of norm one in \( M_p m_p(\Gamma) \).

**Theorem 4.** \( M_p(\Gamma) \) is the algebra of multiplier operators on \( M_p m_p(\Gamma) \).

A weak* compactness argument is used.

\( M_p m_p(\Gamma) \) plays the role in \( M_p(\Gamma) \) that \( L^1(\Gamma) \) plays in \( M(\Gamma) \).

**References**


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