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PROPERTIES OF THREE ALGEBRAS RELATED TO L_p-MULTIPLIERS[1]

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1. **Introduction.** In this paper we shall announce several properties of certain algebras which arise in the study of L_p-multipliers; detailed proofs will be given elsewhere. Let G be a locally compact abelian group and let Γ denote its dual group. Let $L_p(\Gamma)$ denote the space of p-integrable functions on Γ with respect to Haar measure, and let q denote the index which is conjugate to p. Let

$$A_p(\Gamma) = [L_p(\Gamma) \hat{\otimes} L_q(\Gamma)]/K$$

where K is the kernel of the convolution operator $c: L_p \hat{\otimes} L_q(\Gamma) \to C_0(\Gamma)$ by $c(f \otimes g)(\gamma) = (f * g)(\gamma)$. $A_p(\Gamma)$ is the p-Fourier algebra which was introduced by Figa-Talamanca in **[6]** where it was shown that $A_p(\Gamma)^*$ is isometrically isomorphic to $M_p(\Gamma)$, the bounded, translation invariant, linear operators on $L_p(\Gamma)$. Herz **[11]** showed that $A_p(\Gamma)$ is a Banach algebra under pointwise multiplication; it is known that $A_2(\Gamma) = A(\Gamma) = L_1(G)^\wedge$ and that the inclusions $A_2(\Gamma) \subset A_p(\Gamma) \subset A_1(\Gamma) = C_0(\Gamma)$ are norm decreasing if $1 < p < 2$; see **[5]**, **[6]**, **[11]** for the basic properties of $A_p(\Gamma)$. Let $B_p(\Gamma)$ denote the algebra of continuous functions f on Γ such that $f(\gamma)h(\gamma) \in A_p(\Gamma)$ whenever $h \in A_p(\Gamma)$. The multiplier algebra $B_p(\Gamma)$ is a Banach algebra in the operator norm. We have studied $B_p(\Gamma)$ in **[8]**, **[9]**. Fix p in $1 < p < 2$.

Regard $L_1(\Gamma)$ as an algebra of convolution operators on $L_p(\Gamma)$ and let $m_p(\Gamma)$ denote the closure of $L_1(\Gamma)$ in $M_p(\Gamma)$. The first result of this paper says that $B_p(\Gamma)$ is isometrically isomorphic to the dual space $m_p(\Gamma)^*$. In the second result, we use certain properties of $B_p(\Gamma)$ to give a theorem of Eberlein type for $M_p(\Gamma)$. In the final section of the paper, we use

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$m_p(\Gamma)$ to represent $M_p(\Gamma)$ as the multiplier algebra of a certain subalgebra of $M_p(\Gamma)$. For the case when Γ is a compact group, Theorem 1 was proved in **[1]**.

Our work has been influenced by Theorems 1.1 and 1.2 of **[12]**. There McKilligan studied the multiplier algebra B of a commutative, semisimple, Banach algebra A which contains an approximate identity of norm one. A^{**} is equipped with the Arens product $(\circ)$ and B is isometrically embedded in $(A^{**}, \circ)$ by the mapping $T \to T^{**}(j)$ where j is the right identity in A^{**}; see **[2]** for the basic properties of the Arens product. Thus if $T \in B$ and if $\{e_\alpha\}$ is the approximate identity in A, then

$$T^{**}(j)(F) = \lim_\alpha F(T(e_\alpha))$$

for every functional $F \in A^*$.

We shall not distinguish between $M_p(\Gamma)$ and $A_p(\Gamma)^*$. If $H \in L_1(\Gamma)$, let $*H$ denote the corresponding convolution operator on $L_p(\Gamma)$. If $\psi \in m_p(\Gamma)^*$, let $\|\psi\|_*$ denote the norm of ψ. If $h \in A_p(\Gamma)$, $|h|_p$ denotes its norm; if $f \in B_p(\Gamma)$, $|||f|||_p$ is the operator norm; and if $T \in M_p(\Gamma)$, $\|T\|_p$ is the operator (or functional) norm of T. An approximate identity $\{E_\alpha\}$ in $L_1(G)$ which consists of normalized characteristic functions of compact, symmetric neighborhoods of the identity is referred to as a standard approximate identity. The corresponding net $\{\hat{E}_\alpha\}$ in $A_2(\Gamma) = A(\Gamma)$ is also referred to as a standard approximate identity.

After we had submitted this paper for publication, Professor Carl Herz told us that he had given a different proof of Theorem 1 in 1970 for amenable groups. N. Lohoué, C.R. Acad. Sci. Paris **272** (1971) and **273** (1971), refers to Herz's result as presented at Orsay in June, 1970.

2. Dual space representation.

THEOREM 1. *$B_p(\Gamma)$ is isometrically isomorphic to $m_p(\Gamma)^*$ by the map $\varphi \to \tilde{\varphi}$ when $\tilde{\varphi}(*H) = \int_\Gamma \varphi(\gamma)H(\gamma)\, d\gamma$ for all $H \in L_1(\Gamma)$.*

Use Theorems 1 of **[6]** and **[7]** to show that $h \to \tilde{h}$ gives an isometric embedding of $A_p(\Gamma)$ into $m_p(\Gamma)^*$. Use a standard approximate identity to extend this embedding to $B_p(\Gamma)$. Conversely, let $\tilde{\psi} \in m_p(\Gamma)^*$; then there is a bounded measurable function $\psi_0(\gamma)$ such that

$$\tilde{\psi}(*H) = \int_\Gamma \psi_0(\gamma)H(\gamma)\, d\gamma.$$

Define

$$\psi_{\alpha\beta}(\gamma) = (\psi_0 \hat{E}_\alpha) * f_\beta(\gamma)$$

when $\{E_\alpha\}$ and $\{f_\beta\}$ are standard approximate identities in $L_1(G)$ and $L_1(\Gamma)$ respectively. Then $|\psi_{\alpha\beta}|_p \leqq \|\tilde{\psi}\|_*$ and $\{\psi_{\alpha\beta}\}$ converges to ψ_0 in the weak* topology of $L^\infty(\Gamma)$. Let $\mathfrak{B}_p$ denote the algebra of bounded measurable functions ψ on Γ for which $M(\psi)(x, y)=\psi(xy^{-1})$ is a multiplier on $L_p\hat{\otimes}L_q(\Gamma)$. By following Eymard [5], one shows that $\mathfrak{B}_p(\Gamma)=B_p(\Gamma)$. Let $E_q=L_p \otimes_\lambda L_q(\Gamma)$, the completion of $L_p\otimes L_q(\Gamma)$ with respect to the least cross norm. By a theorem of Grothendieck [10, p. 122], $E_q^*= L_p\hat{\otimes}L_q(\Gamma)$. Using this fact one shows that $M(\psi_{\alpha\beta})$ converges to $M(\psi_0)$ in the weak* topology of $L_p\hat{\otimes}L_q(\Gamma)$ and that $\psi_0 \in \mathfrak{B}_p(\Gamma)$.

By letting $m_p(\Gamma)^*=M_p(\Gamma)^*/m_p(\Gamma)^\perp$ have the quotient Arens product, one sees that $\varphi\to\tilde{\varphi}$ is an algebra isomorphism as well.

3. **Eberlein's theorem.** Use McKilligan's representation for multipliers to regard a function $f \in B_p(\Gamma)$ as a functional $\tilde{f} \in M_p(\Gamma)^*$.

THEOREM 2. *Let $M_p(\Gamma)_c$ denote the L_p-multipliers with continuous Fourier transforms. An operator $T \in M_2(\Gamma)_c$ is in $M_p(\Gamma)_c$ if and only if there is a constant $M\geqq 0$ such that for every finite set $\{a_k\}$ of complex numbers and every equinumerous subset $\{g_k\}\subset G$, the Fourier transform $\hat{T}$ of T satisfies*

$$\left|\sum_{k=1}^n a_k\hat{T}(g_k)\right| \leqq M \left|\left|\left|\sum_{k=1}^n a_k\tilde{g}_k\right|\right|\right|_p.$$

When $T \in M_p(\Gamma)_c$, $\|T\|_p$ is the least constant M for which the inequality holds.

If $T \in M_p(\Gamma)_c$, it follows from McKilligan's representation that $\tilde{g}(T)= \hat{T}(g)$ for $g \in G$, so that the inequality holds for some $M\leqq\|T\|_p$. By Saeki's Theorem 4.3 of [14], $\|T\|_p$ is the least constant M for which the inequality holds. If $T \in M_2(\Gamma)_c$ satisfies the inequality, so does $T_{\alpha\beta}= *(\hat{f}_\beta T(E_\alpha))$ when $\{f_\beta\}\subset L_1(G)$ and $\{E_\alpha\}\subset L_1(\Gamma)$ are standard approximate identities. Since $\|T_{\alpha\beta}\|_p\leqq M$, the net $\{T_{\alpha\beta}\}$ has a weak* convergent subnet $\{T_\delta\}$ in $M_p(\Gamma)$. Since $A_2(\Gamma)=A(\Gamma)$ is dense in $A_p(\Gamma)$. it follows that $T= \lim T_\delta$ is in $M_p(\Gamma)$.

From [14], a function $F \in L^\infty(G)$ is said to be regulated if there is an approximate identity $\{E_\alpha\}$ of norm one in $L_1(G)$ such that $\{F * E_\alpha\}$ converges pointwise and in the weak* topology of $L^\infty(G)$ to F. Theorem 2 can be extended so as to apply to operators with regulated Fourier transforms.

The separation theorem [3, p. 417] for compact convex sets and Theorem 2 now imply

THEOREM 3. *If $f \in B_p(\Gamma)$, there is a net $\{f_\beta\}$ in the span of G in $B_p(\Gamma)$ such that $|||f_\beta|||_p \leqq |||f|||_p$ and such that $\{f_\beta\}$ converges to f in the weak* topology of $B_p(\Gamma)$.*

4. M_p **as a multiplier algebra.** Use multiplication of operators to regard $M_p(\Gamma)$ as an algebra over the ring $m_p(\Gamma)$. In particular, $M_p(\Gamma)$ is an $m_p(\Gamma)$-module. It follows from the general form of Cohen's factorization theorem [**13**, p. 453] that the m_p-essential submodule of $M_p(\Gamma)$ is

$$M_p m_p(\Gamma) = \{K \in M_p(\Gamma) \mid K = UT,\ U \in M_p(\Gamma),\ T \in m_p(\Gamma)\}.$$

$M_p m_p(\Gamma)$ is a Banach algebra in the operator norm and a standard approximate identity in $L_1(\Gamma)$ is an approximate identity of norm one in $M_p m_p(\Gamma)$.

THEOREM 4. *$M_p(\Gamma)$ is the algebra of multiplier operators on $M_p m_p(\Gamma)$.*

A weak* compactness argument is used.

$M_p m_p(\Gamma)$ plays the role in $M_p(\Gamma)$ that $L_1(\Gamma)$ plays in $M(\Gamma)$.

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